17

Symmetry and Conservation Laws

Review: Chapter 52, Vol. I, *Symmetry in Physical Laws*

Reference: *Angular Momentum in Quantum Mechanics:* A. R. Edmonds, Princeton University Press, 1957

17–1 Symmetry

In classical physics there are a number of quantities which are *conserved*—such a momentum, energy, and angular momentum. Conservation theorems about corresponding quantities also exist in quantum mechanics. The most beautiful thing of quantum mechanics is that the conservation theorems can, in a sense, be derived from something else, whereas in classical mechanics they are practically the starting points of the laws. (There are ways in classical mechanics to do an analogous thing to what we will do in quantum mechanics, but it can be done only at a very advanced level.) In quantum mechanics, however, the conservation laws are very deeply related to the principle of superposition of amplitudes, and to the symmetry of physical systems under various changes. This is the subject of the present chapter. Although we will apply these ideas mostly to the conservation of angular momentum, the essential point is that the theorems about the conservation of all kinds of quantities are—in the quantum mechanics—related to the symmetries of the system.

We begin, therefore, by studying the question of symmetries of systems. A very simple example is the hydrogen molecular ion—we could equally well take the ammonia molecule—in which there are two states. For the hydrogen molecular ion we took as our base states one in which the electron was located near proton number 1, and another in which the electron was located near proton number 2. The two states—which we called $|\,1\rangle$ and $|\,2\rangle$—are shown again in Fig. 17–1(a). Now, so long as the two nuclei are both exactly the same, then there is a certain *symmetry* in this physical system. That is to say, if we were to *reflect* the system in the plane halfway between the two protons—by which we mean that everything on one side of the plane gets moved to the symmetric position on the other side—we would get the situations in Fig. 17–1(b). Since the protons are identical, the *operation of reflection* changes $|\,1\rangle$ into $|\,2\rangle$ and $|\,2\rangle$ into $|\,1\rangle$. We'll call this reflection operation $\hat{P}$ and write

$$\hat{P}\,|\,1\rangle = |\,2\rangle, \qquad \hat{P}\,|\,2\rangle = |\,1\rangle. \tag{17.1}$$

So our $\hat{P}$ is an operator in the sense that it "*does something*" to a state to make a new state. The interesting thing is that $\hat{P}$ operating on *any* state produces some *other* state of the system.

Now $\hat{P}$, like any of the other operators we have described, has matrix elements which can be defined by the usual obvious notation. Namely,

$$P_{11} = \langle 1\,|\,\hat{P}\,|\,1\rangle \qquad \text{and} \qquad P_{12} = \langle 1\,|\,\hat{P}\,|\,2\rangle$$

are the matrix elements we get if we multiply $\hat{P}\,|\,1\rangle$ and $\hat{P}\,|\,2\rangle$ on the left by $\langle 1\,|$. From Eq. (17.1) they are

$$\begin{aligned} \langle 1\,|\,\hat{P}\,|\,1\rangle &= P_{11} = \langle 1\,|\,2\rangle = 0,\\ \langle 1\,|\,\hat{P}\,|\,2\rangle &= P_{12} = \langle 1\,|\,1\rangle = 1. \end{aligned} \tag{17.2}$$

In the same way we can get P_{21} and P_{22}. The matrix of $\hat{P}$—*with respect to the base system* $|\,1\rangle$ *and* $|\,2\rangle$—is

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{17.3}$$

We see once again that the words *operator* and *matrix* in quantum mechanics are

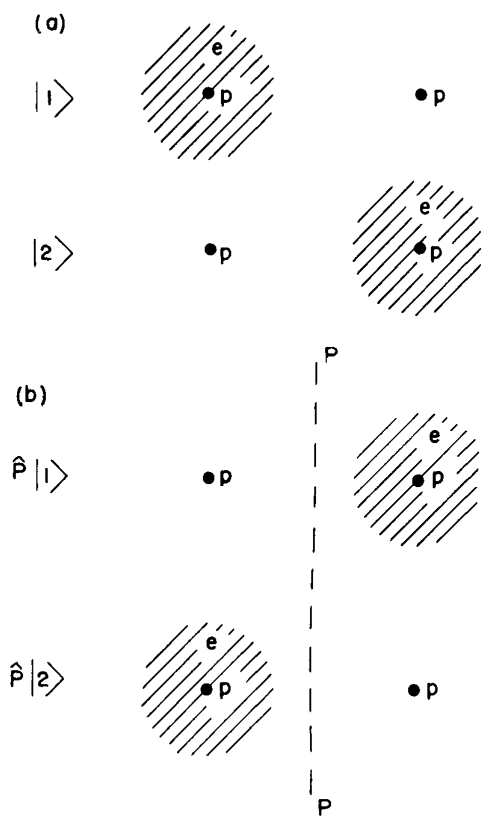

Fig. 17–1. If the states $|\,1\rangle$ and $|\,2\rangle$ are reflected in the plane *P-P*, they go into $|\,2\rangle$ and $|\,1\rangle$, respectively.

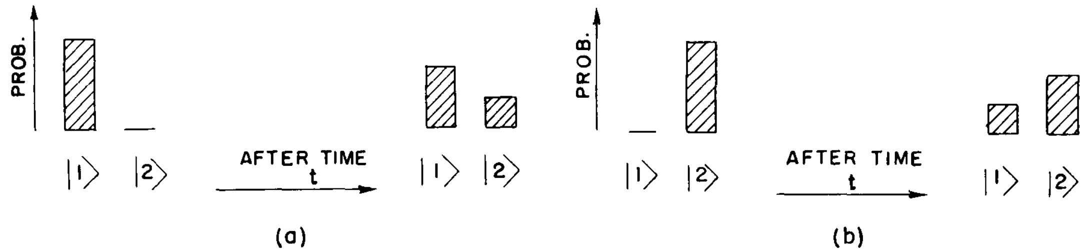

Fig. 17–2. In a symmetric system, if a pure $|\,1\rangle$ state develops as shown in part (a), a pure $|\,2\rangle$ state will develop as in part (b).

practically interchangeable. There are slight technical differences—like the difference between a "numeral" and a "number"—but the distinction is something pedantic that we don't have to worry about. So whether $\hat{P}$ defines an operation, or is actually used to define a matrix of numbers, we will call it interchangeably an operator or a matrix.

Now we would like to point out something. We will *suppose* that the *physics* of the whole hydrogen molecular ion system is *symmetrical.* It doesn't have to be —it depends, for instance, on what else is near it. But if the system is symmetrical, the following idea should certainly be true. Suppose we start at $t = 0$ with the system in the state $|\,1\rangle$ and find after an interval of time t that the system turns out to be in a more complicated situation—in some linear combination of the two base states. Remember that in Chapter 8 we used to represent "going for a period of time" by multiplying by the operator $\hat{U}$. That means that the system would after a while—say 15 seconds to be definite—be in some other state. For example, it might be $\sqrt{2/3}$ parts of the state $|\,1\rangle$ and $i\sqrt{1/3}$ parts of the state $|\,2\rangle$, and we would write

$$|\,\psi \text{ at } 15 \text{ sec}\rangle = \hat{U}(15, 0)\,|\,1\rangle = \sqrt{2/3}\,|\,1\rangle + i\sqrt{1/3}\,|\,2\rangle. \tag{17.4}$$

Now we ask what happens if we start the system in the *symmetric* state $|\,2\rangle$ and wait for 15 seconds under the *same conditions*? It is clear that if the world is symmetric—as we are supposing—we should get the state symmetric to (17.4):

$$|\,\psi \text{ at } 15 \text{ sec}\rangle = \hat{U}(15, 0)\,|\,2\rangle = \sqrt{2/3}\,|\,2\rangle + i\sqrt{1/3}\,|\,1\rangle. \tag{17.5}$$

The same ideas are sketched diagrammatically in Fig. 17–2. So if the *physics* of a system is symmetrical with respect to some plane, and we work out the behavior of a particular state, we also know the behavior of the state we would get by reflecting the original state in the symmetry plane.

We would like to say the same things a litte bit more generally—which means a little more abstractly. Let $\hat{Q}$ be any one of a number of operations that you could perform on a system *without changing the physics.* For instance, for $\hat{Q}$ we might be thinking of $\hat{P}$, the operation of a *reflection* in the plane between the two atoms in the hydrogen molecule. Or, in a system with two electrons, we might be thinking of the operation of *interchanging* the two electrons. Another possibility would be, in a spherically symmetric system, the operation of a *rotation* of the whole system through a finite angle around some axis—which wouldn't change the physics. Of course, we would normally want to give each special case some special notation for $\hat{Q}$. Specifically, we will normally define the $\hat{R}_y(\theta)$ to be the operation "rotate the system about the y-axis by the angle θ". By $\hat{Q}$ we mean just any one of the operators we have described or any other one—which leaves the basic physical situation unchanged.

Let's think of some more examples. If we have an atom with *no external magnetic* field or *no external electric* field, and if we were to turn the coordinates around any axis, it would be the same physical system. Again, the ammonia molecule is symmetrical with respect to a reflection in a plane parallel to that of the three hydrogens—so long as there is no electric field. When there is an electric field, when we make a reflection we would have to change the electric field also, and that changes the physical problem. But if we have no external field, the molecule is symmetrical.

Now we consider a general situation. Suppose we start with the state $|\psi_1\rangle$ and after some time or other under given physical conditions it has become the state $|\psi_2\rangle$. We can write

$$|\psi_2\rangle = \hat{U}\,|\psi_1\rangle. \tag{17.6}$$

[You can be thinking of Eq. (17.4).] Now imagine we perform the operation $\hat{Q}$ on the whole system. The state $|\psi_1\rangle$ will be transformed to a state $|\psi_1'\rangle$, which we can also write as $\hat{Q}\,|\psi_1\rangle$. Also the state $|\psi_2\rangle$ is changed into $|\psi_2'\rangle = \hat{Q}\,|\psi_2\rangle$. Now *if the physics* is symmetrical under $\hat{Q}$ (don't forget the *if*; it is not a general property of systems), then, waiting for the same time under the same conditions, we should have

$$|\psi_2'\rangle = \hat{U}\,|\psi_1'\rangle. \tag{17.7}$$

[Like Eq. (17.5).] But we can write $\hat{Q}\,|\psi_1\rangle$ for $|\psi_1'\rangle$ and $\hat{Q}\,|\psi_2\rangle$ for $|\psi_2'\rangle$ so (17.7) can also be written

$$\hat{Q}\,|\psi_2\rangle = \hat{U}\hat{Q}\,|\psi_1\rangle. \tag{17.8}$$

If we now replace $|\psi_2\rangle$ by $\hat{U}\,|\psi_1\rangle$—Eq. (17.6)—we get

$$\hat{Q}\hat{U}\,|\psi_1\rangle = \hat{U}\hat{Q}\,|\psi_1\rangle. \tag{17.9}$$

It's not hard to understand what this means. Thinking of the hydrogen ion it says that: "making a reflection and waiting a while"—the expression on the right of Eq. (17.9)—is the same as "waiting a while and then making a reflection"—the expression on the left of (17.9). These should be the same so long as U doesn't change under the reflection.

Since (17.9) is true for *any* starting state $|\psi_1\rangle$, it is really an equation about the operators:

$$\hat{Q}\hat{U} = \hat{U}\hat{Q}. \tag{17.10}$$

This is what we wanted to get—*it is a mathematical statement of symmetry.* When Eq. (17.10) is true, we say that the operators $\hat{U}$ and $\hat{Q}$ *commute.* We can then *define* "symmetry" in the following way: A physical system is *symmetric* with respect to the operation $\hat{Q}$ when $\hat{Q}$ commutes with $\hat{U}$, the operation of the passage of time. [In terms of matrices, the product of two operators is equivalent to the matrix product, so Eq. (17.10) also holds for the matrices Q and U for a system which is symmetric under the transformation Q.]

Incidentally, since for infinitesimal times ϵ we have $\hat{U} = 1 - i\hat{H}\epsilon/\hbar$—where $\hat{H}$ is the usual Hamiltonian (see Chapter 8)—you can see that if (17.10) is true, it is also true that

$$\hat{Q}\hat{H} = \hat{H}\hat{Q}. \tag{17.11}$$

So (17.11) is the mathematical statement of the condition for the symmetry of a physical situation under the operator $\hat{Q}$. It *defines* a symmetry.

17-2 Symmetry and conservation

Before applying the result we have just found, we would like to discuss the idea of symmetry a little more. Suppose that we have a very special situation: after we operate on a state with $\hat{Q}$, we get the same state. This is a very special case, but let's suppose it happens to be true for a state $|\psi_0\rangle$ that $|\psi'\rangle = \hat{Q}\,|\psi_0\rangle$ is physically the same state as $|\psi_0\rangle$. That means that $|\psi'\rangle$ is equal to $|\psi_0\rangle$ except for some phase factor.† How can that happen? For instance, suppose that we

† Incidentally, you can show that $\hat{Q}$ is necessarily a *unitary operator*—which means that if it operates on $|\psi\rangle$ to give some number times $|\psi\rangle$, the number must be of the form $e^{i\delta}$, where δ is real. It's a small point, and the proof rests on the following observation. Any operation like a reflection or a rotation doesn't lose any particles, so the normalization of $|\psi'\rangle$ and $|\psi\rangle$ must be the same; they can only differ by a pure imaginary phase factor.

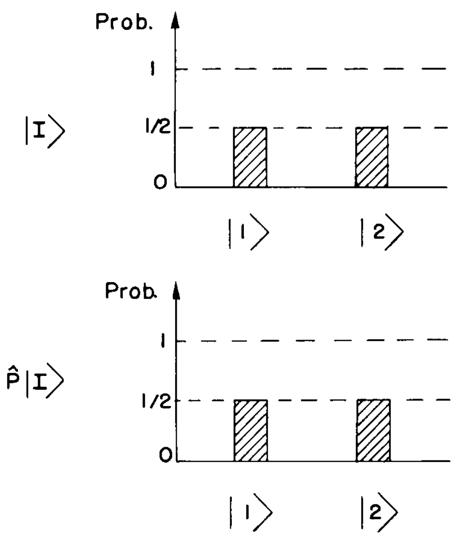

Fig. 17–3. The state $|\,I\rangle$ and the state $\hat{P}\,|\,I\rangle$ obtained by reflecting $|\,I\rangle$ in the central plane.

have an H_2^+ ion in the state which we once called $|\,I\rangle$. For this state there is equal amplitude to be in the base states $|\,1\rangle$ and $|\,2\rangle$. The probabilities are shown as a bar graph in Fig. 17–3(a). If we operate on $|\,I\rangle$ with the reflection operator $\hat{P}$, it flips the state over changing $|\,1\rangle$ to $|\,2\rangle$ and $|2\rangle$ to $|\,1\rangle$—we get the probabilities shown in Fig. 17–3(b). But that's just the state $|\,I\rangle$ all over again. If we start with state $|\,II\rangle$ the probabilities before and after reflection look just the same. However, there is a difference if we look at the *amplitudes*. For the state $|\,I\rangle$ the amplitudes are the *same* after the reflection, but for the state $|\,II\rangle$ the amplitudes have the opposite sign. In other words,

$$\begin{aligned}\hat{P}\,|\,I\rangle &= \hat{P}\left\{\frac{|\,1\rangle + |\,2\rangle}{\sqrt{2}}\right\} = \frac{|\,2\rangle + |\,1\rangle}{\sqrt{2}} = |\,I\rangle,\\ \hat{P}\,|\,II\rangle &= \hat{P}\left\{\frac{|\,1\rangle - |\,2\rangle}{\sqrt{2}}\right\} = \frac{|\,2\rangle - |\,1\rangle}{\sqrt{2}} = -\,|\,II\rangle.\end{aligned} \tag{17.12}$$

If we write $\hat{P}\,|\,\psi_0\rangle = e^{i\delta}\,|\,\psi_0\rangle$, we have that $e^{i\delta} = 1$ for the state $|\,I\rangle$ and $e^{i\delta} = -1$ for the state $|\,II\rangle$.

Let's look at another example. Suppose we have a RHC polarized photon propagating in the z-direction. If we do the operation of a rotation around the z-axis, we know that this just multiplies the amplitude by $e^{i\phi}$ when ϕ is the angle of the rotation. So for the rotation operation in this case, δ is just equal to the angle of rotation.

Now it is clear that *if it happens to be true* that an operator $\hat{Q}$ just changes the phase of a state at some time, say $t = 0$, *it is true forever*. In other words, if the state $|\,\psi_1\rangle$ goes over into the state $|\,\psi_2\rangle$ after a time t, or

$$\hat{U}(t, 0)\,|\,\psi_1\rangle = |\,\psi_2\rangle \tag{17.13}$$

and if the symmetry of the situation makes it so that

$$\hat{Q}\,|\,\psi_1\rangle = e^{i\delta}\,|\,\psi_1\rangle, \tag{17.14}$$

then it is also that

$$\hat{Q}\,|\,\psi_2\rangle = e^{i\delta}\,|\,\psi_2\rangle. \tag{17.15}$$

This is clear, since

$$\hat{Q}\,|\,\psi_2\rangle = \hat{Q}\hat{U}\,|\,\psi_1\rangle = \hat{U}\hat{Q}\,|\,\psi_1\rangle,$$

and if $\hat{Q}\,|\,\psi_1\rangle = e^{i\delta}\,|\,\psi_1\rangle$, then

$$\hat{Q}\,|\,\psi_2\rangle = \hat{U}e^{i\delta}\,|\,\psi_1\rangle = e^{i\delta}\hat{U}\,|\,\psi_1\rangle = e^{i\delta}\,|\,\psi_2\rangle.$$

[The sequence of equalities follows from (17.13) and (17.10) for a symmetrical system, from (17.14), and from the fact that a number like $e^{i\delta}$ commutes with an operator.]

So with certain symmetries something which is true initially is true for all times. But isn't that just a *conservation law*? Yes! It says that if you look at the original state and by making a little computation on the side discover that an operation *which is a symmetry operation of the system* produces only a multiplication by a certain phase, then you know that the same property will be true of the final state—the same operation multiplies the final state by the same phase factor. This is always true even though we may not know anything else about the inner mechanism of the universe which changes a system from the initial to the final state. Even if we do not care to look at the details of the machinery by which the system gets from one state to another, we can still say that if a thing is in a state with a certain symmetry character originally, and if the Hamiltonian for this thing is symmetrical under that symmetry operation, then the state will have the same symmetry character for all times. That's the basis of all the conservation laws of quantum mechanics.

Let's look at a special example. Let's go back to the $\hat{P}$ operator. We would like first to modify a little our definition of $\hat{P}$. We want to take for $\hat{P}$ not just a mirror reflection, because that requires defining the plane in which we put the mirror. There is a special kind of a reflection that doesn't require the specification of a plane. Suppose we redefine the operation $\hat{P}$ this way: First you reflect in a mirror in the z-plane so that z goes to $-z$, x stays x, and y stays y; then you turn the system 180° about the z-axis so that x is made to go to $-x$ and y to $-y$. The whole thing is called an *inversion.* Every point is projected *through the origin* to the diametrically opposite position. All the coordinates of everything are reversed. We will still use the symbol $\hat{P}$ for this operation. It is shown in Fig. 17–4. It is a little more convenient than a simple reflection because it doesn't require that you specify which coordinate plane you used for the reflection—you need specify only the point which is at the center of symmetry.

Now let's suppose that we have a state $|\,\psi_0\rangle$ which under the inversion operation goes into $e^{i\delta}\,|\,\psi_0\rangle$—that is,

$$|\,\psi_0'\rangle = \hat{P}\,|\,\psi_0 = e^{i\delta}\,|\,\psi_0\rangle. \tag{17.16}$$

Then suppose that we invert again. After *two* inversions we are right back where we started from—nothing is changed at all. We must have that

$$\hat{P}\,|\,\psi_0'\rangle = \hat{P}\cdot\hat{P}\,|\,\psi_0\rangle = |\,\psi_0\rangle.$$

But

$$\hat{P}\cdot\hat{P}\,|\,\psi_0\rangle = \hat{P}e^{i\delta}\,|\,\psi_0\rangle = e^{i\delta}\,\hat{P}\,|\,\psi_0\rangle = (e^{i\delta})^2\,|\,\psi_0\rangle.$$

It follows that

$$(e^{i\delta})^2 = 1.$$

So *if the inversion operator is a symmetry operation* of a state, there are only two possibilities for δ:

$$e^{i\delta} = \pm 1,$$

which means that

$$\hat{P}\,|\,\psi_0\rangle = |\,\psi_0\rangle \qquad \text{or} \qquad \hat{P}\,|\,\psi_0\rangle = -\,|\,\psi_0\rangle. \tag{17.17}$$

Classically, if a state is symmetric under an inversion, the operation gives back the same state. In quantum mechanics, however, there are the two possibilities: we get the *same state* or *minus* the same state. When we get the *same* state, $\hat{P}\,|\,\psi_0\rangle = |\,\psi_0\rangle$, we say that the state $|\,\psi_0\rangle$ has *even parity.* When the sign is reversed so that $\hat{P}\,|\,\psi_0\rangle = -\,|\,\psi_0\rangle$, we say that the state has *odd parity.* (The inversion operator $\hat{P}$ is also known as the parity operator.) The state $|\,I\rangle$ of the H_2^+ ion has even parity; and the state $|\,II\rangle$ has odd parity—see Eq. (17.12). There are, of course, states which are not symmetric under the operation $\hat{P}$; these are states with no definite parity. For instance, in the H_2^+ system the state $|\,I\rangle$ has even parity, the state $|\,II\rangle$ has odd parity, and the state $|\,1\rangle$ has no definite parity.

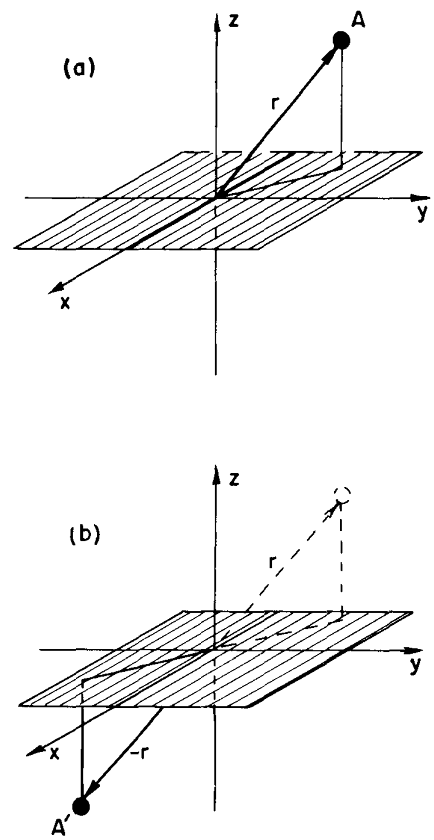

Fig. 17–4. The operation of inversion, $\hat{P}$. Whatever is at the point A at (x, y, z) is moved to the point A' at $(-x, -y, -z)$.

When we speak of an operation like inversion being performed "*on a physical system*" we can think about it in two ways. We can think of *physically moving* whatever is at $\boldsymbol{r}$ to the inverse point at $-\boldsymbol{r}$, or we can think of *looking* at the same system from a new frame of reference x', y', z' related to the old by $x' = -x$, $y' = -y$, and $z' = -z$. Similarly, when we think of rotations, we can think of rotating bodily a physical system, or of rotating the coordinate frame with respect to which we measure the system, keeping the "system" fixed in space. Generally, the two points of view are essentially equivalent. For rotation they are equivalent *except* that rotating a *system* by the angle θ is like rotating the reference frame by the *negative* of θ. In these lectures we have usually considered what happens when a projection is made into a new set of axes. What you get that way is the same as what you get if you leave the axes fixed and rotate the system *backwards* by the same amount. When you do that, the signs of the angles are reversed.†

† In other books you may find formulas with different signs; they are probably using a different definition of the angles.

Many of the *laws* of physics—but not all—are unchanged by a reflection or an inversion of the coordinates. They are *symmetric* with respect to an inversion. The laws of electrodynamics, for instance, are unchanged if we change x to $-x$, y to $-y$, and z to $-z$ in *all* the equations. The same is true for the laws of gravity, and for the strong interactions of nuclear physics. Only the weak interactions—responsible for β-decay—do not have this symmetry. (We discussed this in some detail in Chapter 52, Vol. I.) We will for now leave out any consideration of the β-decays. Then in any physical system where β-decays are not expected to produce any appreciable effect—an example would be the emission of light by an atom—the Hamiltonian $\hat{H}$ and the operator $\hat{P}$ will commute. Under these circumstances we have the following proposition. If a state originally has even parity, and if you look at the physical situation at some later time, it will again have even parity. For instance, suppose an atom about to emit a photon is in a state known to have even parity. You look at the whole thing—including the photon—after the emission; it will again have even parity (likewise if you start with odd parity). This principle is called the *conservation of parity*. You can see why the words "conservation of parity" and "reflection symmetry" are closely intertwined in the quantum mechanics. Although until a few years ago it was thought that nature always conserved parity, it is now known that this is *not* true. It has been discovered to be false because the β-decay reaction does not have the inversion symmetry which is found in the other laws of physics.

Now we can prove an interesting theorem (which is true so long as we can disregard weak interactions): Any state of definite energy which is not degenerate must have a definite parity. It must have either even parity or odd parity. (Remember that we have sometimes seen systems in which several states have the same energy—we say that such states are *degenerate*. Our theorem will not apply to them.)

For a state $|\,\psi_0\rangle$ of definite energy, we know that

$$\hat{H}\,|\,\psi_0\rangle = E\,|\,\psi_0\rangle, \tag{17.18}$$

where E is just a number—the energy of the state. If we have *any* operator $\hat{Q}$ which is a symmetry operator of the system we can prove that

$$\hat{Q}\,|\,\psi_0\rangle = e^{i\delta}\,|\,\psi_0\rangle \tag{17.19}$$

so long as $|\,\psi_0\rangle$ is a unique state of definite energy. Consider the new state $|\,\psi_0'\rangle$ that you get from operating with $\hat{Q}$. If the physics is symmetric, then $|\,\psi_0'\rangle$ must have the same energy as $|\,\psi_0\rangle$. But we have taken a situation in which there *is only one* state of that energy, namely $|\,\psi_0\rangle$, so $|\,\psi_0'\rangle$ must be the same state—it can only differ by a phase. That's the physical argument.

The same thing comes out of our mathematics. Our definition of symmetry is Eq. (17.10) or Eq. (17.11) (good for any state ψ),

$$\hat{H}\hat{Q}\,|\,\psi\rangle = \hat{Q}\hat{H}\,|\,\psi\rangle. \tag{17.20}$$

But we are considering only a state $|\,\psi_0\rangle$ which is a definite energy state, so that $\hat{H}\,|\,\psi_0\rangle = E\,|\,\psi_0\rangle$. Since E is just a number that floats through $\hat{Q}$ if we want, we have

$$\hat{Q}\hat{H}\,|\,\psi_0\rangle = \hat{Q}E\,|\,\psi_0\rangle = E\hat{Q}\,|\,\psi_0\rangle.$$

So

$$\hat{H}\{\hat{Q}\,|\,\psi_0\rangle\} = E\{\hat{Q}\,|\,\psi_0\rangle\}. \tag{17.21}$$

So $|\,\psi_0'\rangle = \hat{Q}\,|\,\psi_0\rangle$ is also a definite energy state of $\hat{H}$—and with the same E. But by our hypothesis, there is only one such state; it must be that $|\,\psi_0'\rangle = e^{i\delta}\,|\,\psi_0\rangle$.

What we have just proved is true for any operator $\hat{Q}$ that is a symmetry operator of the physical system. Therefore, in a situation in which we consider only electrical forces and strong interactions—and no β-decay—so that inversion symmetry is an allowed approximation, we have that $\hat{P}\,|\,\psi\rangle = e^{i\delta}\,|\,\psi\rangle$. But we have also seen that $e^{i\delta}$ must be either $+1$ or -1. So any state of a definite energy (which is not degenerate) has got either an even parity or an odd parity.

17–3 The conservation laws

We turn now to another interesting example of an operation: a rotation. We consider the special case of an operator that rotates an atomic system by angle ϕ around the z-axis. We will call this operator† $\hat{R}_z(\phi)$. We are going to suppose that we have a physical situation where we have no influences lined up along the x- and y-axes. Any electric field or magnetic field is taken to be parallel to the z-axis‡ so that there will be no change in the *external* conditions if we rotate the whole physical system about the z-axis. For example, if we have an atom in empty space and we turn the atom around the z-axis by an angle ϕ, we have the same physical system.

Now then, there are *special states* which have the property that such an operation produces a new state which is the original state multiplied by some phase factor. Let us make a quick side remark to show you that when this is true the phase change must always be proportional to the angle ϕ. Suppose that you would rotate twice by the angle ϕ. That's the same thing as rotating by the angle 2ϕ. If a rotation by ϕ has the effect of multiplying the state $|\,\psi_0\rangle$ by a phase $e^{i\delta}$ so that

$$\hat{R}_z(\phi)\,|\,\psi_0\rangle = e^{i\delta}\,|\,\psi_0\rangle,$$

two such rotations in succession would multiply the state by the factor $(e^{i\delta})^2 = e^{i2\delta}$, since

$$\hat{R}_z(\phi)\hat{R}_z(\phi)\,|\,\psi_0\rangle = \hat{R}_z(\phi)e^{i\delta}\,|\,\psi_0\rangle = e^{i\delta}\hat{R}_z(\phi)\,|\,\psi_0\rangle = e^{i\delta}e^{i\delta}\,|\,\psi_0\rangle.$$

The phase change δ must be proportional to ϕ.¶ We are considering then those special states $|\,\psi_0\rangle$ for which

$$\hat{R}_z(\phi)\,|\,\psi_0\rangle = e^{im\phi}\,|\,\psi_0\rangle, \tag{17.22}$$

where m is some real number.

We also know the remarkable fact that *if* the system is symmetrical for a rotation around z *and if* the original state happens to have the property that (17.22) is true, then it will also have the same property later on. So this number m is a very important one. If we know its value initially, we know its value at the end of the game. It is a number which is *conserved—m* is a *constant of the motion.* The reason that we pull out m is because it hasn't anything to do with any special angle ϕ, and also because it corresponds to something in classical mechanics. In *quantum* mechanics we *choose* to call $m\hbar$—for such states as $|\,\psi_0\rangle$—the *angular momentum about the z-axis.* If we do that we find that in the limit of large systems the same quantity is equal to the z-component of the angular momentum of classical mechanics. So if we have a state for which a rotation about the z-axis just produces a phase factor $e^{im\phi}$, then we have a state of definite angular momentum about that axis—and the angular momentum is conserved. It is $m\hbar$ now and forever. Of course, you can rotate about any axis, and you get the conservation of angular momentum for the various axes. You see that the conservation of angular momentum is related to the fact that when you turn a system you get the same state with only a new phase factor.

We would like to show you how general this idea is. We will apply it to two other conservation laws which have exact correspondence in the physical ideas to the conservation of angular momentum. In classical physics we also have conservation of momentum and conservation of energy, and it is interesting to see that both of these are related in the same way to some physical symmetry.

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† Very precisely, we will define $\hat{R}_z(\phi)$ as a rotation of the physical system by $-\phi$ about the z-axis, which is the same as rotating the coordinate frame by $+\phi$.

‡ We can always choose z along the direction of the field provided there is only one field at a time, and its direction doesn't change.

¶ For a fancier proof we should make this argument for small rotations ϵ. Since any angle ϕ is the sum of a suitable n number of these, $\phi = n\epsilon$, $\hat{R}_z(\phi) = [\hat{R}_z(\epsilon)]^n$ and the total phase change is n times that for the small angle ϵ, and is, therefore, proportional to ϕ

Suppose that we have a physical system—an atom, some complicated nucleus, or a molecule, or something—and it doesn't make any difference if we take the whole system and move it over to a different place. So we have a Hamiltonian which has the property that it depends only on the *internal coordinates* in some sense, and does not depend on the *absolute position* in space. Under those circumstances there is a special symmetry operation we can perform which is a translation in space. Let's define $\hat{D}_x(a)$ as the operation of a displacement by the distance a along the x-axis. Then for any state we can make this operation and get a new state. But again there can be very special states which have the property that when you displace them by a along the x-axis you get the same state except for a phase factor. It's also possible to prove, just as we did above, that when this happens, the phase must be proportional to a. So we can write for these special states $|\,\psi_0\rangle$

$$\hat{D}_x(a)\,|\,\psi_0\rangle = e^{ika}\,|\,\psi_0\rangle. \tag{17.23}$$

The coefficient k, when multiplied by $\hbar$, is called *the x-component of the momentum.* And the reason it is called that is that this number is numerically equal to the classical momentum p_x when we have a large system. The general statement is this: If the Hamiltonian is unchanged when the system is displaced, and if the state starts with a definite momentum in the x-direction, then the momentum in the x-direction will remain the same as time goes on. The total momentum of a system before and after collisions—or after explosions or what not—will be the same.

There is another operation that is quite analogous to the displacement in space: a delay in time. Suppose that we have a physical situation where there is *nothing external* that depends on time, and we start something off at a certain moment in a given state and let it roll. Now if we were to start the same thing off again (in another experiment) two seconds later—or/say, delayed by a time τ—and if nothing in the external conditions depends on the absolute time, the development would be the same and the final state would be the same as the other final state, except that it will get there later by the time τ. Under those circumstances we can also find special states which have the property that the development in time has the special characteristic that the delayed state is just the old, multiplied by a phase factor. Once more it is clear that for these special states the phase change must be proportional to τ. We can write

$$\hat{D}_t(\tau)\,|\,\psi_0\rangle = e^{-i\omega\tau}\,|\,\psi_0\rangle. \tag{17.24}$$

It is conventional to use the negative sign in defining ω; with this convention $\omega\hbar$ is the *energy* of the system, *and it is conserved.* So a system of definite energy is one which when displaced τ in time reproduces itself multiplied by $e^{-i\omega\tau}$. (That's what we have said before when we defined a quantum state of definite energy, so we're consistent with ourselves.) It means that if a system is in a state of definite energy, and if the Hamiltonian doesn't depend on t, then no matter what goes on, the system will have the same energy at all later times.

You see, therefore, the relation between the conservation laws and the symmetry of the world. Symmetry with respect to displacements in time implies the conservation of energy; symmetry with respect to position in x, y, or z implies the conservation of that component of momentum. Symmetry with respect to rotations around the x-, y-, and z-axes implies the conservation of the x-, y-, and z-components of angular momentum. Symmetry with respect to reflection implies the conservation of parity. Symmetry with respect to the interchange of two electrons implies the conservation of something we don't have a name for, and so on. Some of these principles have classical analogs and others do not. There are more conservation laws in quantum mechanics than are useful in classical mechanics—or, at least, than are usually made use of.

In order that you will be able to read other books on quantum mechanics, we must make a small technical aside—to describe the notation that people use. The operation of a displacement with respect to time is, of course, just the operation $\hat{U}$ that we talked about before:

$$\hat{D}_t(\tau) = \hat{U}(t + \tau, t). \tag{17.25}$$

Most people like to discuss everything in terms of *infinitesimal* displacements in time, or in terms of infinitesimal displacements in space, or in terms of rotations through infinitesimal angles. Since any finite displacement or angle can be accumulated by a succession of infinitesimal displacements or angles, it is often easier to analyze first the infinitesimal case. The operator of an infinitesimal displacement Δt in time is—as we have defined it in Chapter 8—

$$\hat{D}_t(\Delta t) = 1 - \frac{i}{\hbar}\,\Delta t\hat{H}. \tag{17.26}$$

Then $\hat{H}$ is analogous to the classical quantity we call energy, because if $\hat{H}\,|\,\psi\rangle$ happens to be a constant times $|\,\psi\rangle$ namely, $\hat{H}\,|\,\psi\rangle = E\,|\,\psi\rangle$, then that constant is the energy of the system.

The same thing is done for the other operations. If we make a small displacement in x, say by the amount Δx, a state $|\,\psi\rangle$ will, *in general*, go over into some other state $|\,\psi'\rangle$. We can write

$$|\,\psi'\rangle = \hat{D}_x(\Delta x)\,|\,\psi\rangle = \left(1 + \frac{i}{\hbar}\,\hat{p}_x\,\Delta x\right)|\,\psi\rangle, \tag{17.27}$$

since as Δx goes to zero, the $|\,\psi'\rangle$ should become just $|\,\psi\rangle$ or $\hat{D}_x(0) = 1$, and for small Δx the change of $\hat{D}_x(\Delta x)$ from 1 should be proportional to Δx. Defined this way, the operator $\hat{p}_x$ is called the momentum operator—for the x-component, of course.

For identical reasons, people usually write for small rotations

$$\hat{R}_z(\Delta\phi)\,|\,\psi\rangle = \left(1 + \frac{i}{\hbar}\,\hat{J}_z\,\Delta\phi\right)|\,\psi\rangle \tag{17.28}$$

and call $\hat{J}_z$ the operator of the z-component of angular momentum. For those special states for which $\hat{R}_z(\phi)\,|\,\psi_0\rangle = e^{im\phi}\,|\,\psi_0\rangle$, we can for any small angle—say $\Delta\phi$—expand the right-hand side to first order in $\Delta\phi$ and get

$$\hat{R}_z(\Delta\phi) = e^{im\Delta\phi}\,|\,\psi_0\rangle = (1 + im\Delta\phi)\,|\,\psi_0\rangle.$$

Comparing this with the definition of $\hat{J}_z$ in Eq. (17.28), we get that

$$\hat{J}_z\,|\,\psi_0\rangle = m\hbar\,|\,\psi_0\rangle. \tag{17.29}$$

In other words, if you operate with $\hat{J}_z$ on a state *with a definite angular momentum* about the z-axis, you get $m\hbar$ times the same state, where $m\hbar$ is the amount of z-component of angular momentum. It is quite analogous to operating on a definite energy state with $\hat{H}$ to get $E\,|\,\psi\rangle$.

We would now like to make some applications of the ideas of the conservation of angular momentum—to show you how they work. The point is that they are really very simple. You knew before that angular momentum is conserved. The only thing you really have to remember from this chapter is that if a state $|\,\psi_0\rangle$ has the property that upon a rotation through an angle ϕ about the z-axis, it becomes $e^{im\phi}\,|\,\psi_0\rangle$; it has a z-component of angular momentum equal to $m\hbar$. That's all we will need to do a number of interesting things.

17–4 Polarized light

First of all we would like to check on one idea. In Section 11–4 we showed that when RHC polarized light is viewed in a frame rotated by the angle ϕ about the z-axis† it gets multiplied by $e^{i\phi}$. Does that mean then that the photons of light

† Sorry! This angle is the negative of the one we used in Section 11–4.

that are right circularly polarized carry an angular momentum of *one* unit† along the z-axis? *Indeed it does.* It also means that if we have a beam of light containing a large number of photons all circularly polarized the same way—as we would have in a classical beam—it will carry angular momentum. If the total energy carried by the beam in a certain time is W, then there are $N = W/\hbar\omega$ photons. Each one carries the angular momentum $\hbar$, so there is a total angular momentum of

$$J_z = N\hbar = \frac{W}{\omega}. \tag{17.30}$$

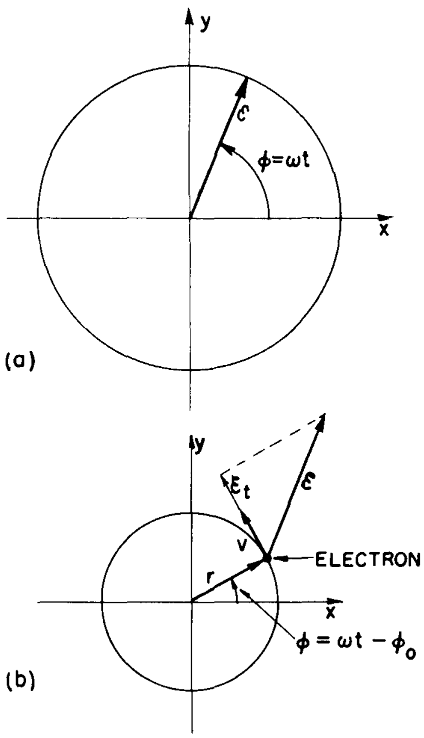

Fig. 17–5. (a) The electric field $\mathcal{E}$ in a circularly polarized light wave. (b) The motion of an electron being driven by the circularly polarized light.

Can we prove classically that light which is right circularly polarized carries an energy and angular momentum in proportion to W/ω? That should be a classical proposition if everything is right. Here we have a case where we can go from the quantum thing to the classical thing. We should see if the classical physics checks. It will give us an idea whether we have a right to call m the angular momentum. Remember what right circularly polarized light is, classically. It's described by an electric field with an oscillating x-component and an oscillating y-component 90° out of phase so that the resultant electric vector $\boldsymbol{\mathcal{E}}$ goes in a circle—as drawn in Fig. 17–5(a). Now suppose that such light shines on a wall which is going to absorb it—or at least some of it—and consider an atom in the wall according to the classical physics. We have often described the motion of the electron in the atom as a harmonic oscillator which can be driven into oscillation by an external electric field. We'll suppose that the atom is isotropic, so that it can oscillate equally well in the x- or y-directions. Then in the circularly polarized light, the x-displacement and the y-displacement are the same, but one is 90° behind the other. The net result is that the electron moves in a circle, as shown in Fig. 17–5(b). The electron is displaced at some displacement $\boldsymbol{r}$ from its equilibrium position at the origin and goes around with some phase lag with respect to the vector $\boldsymbol{\mathcal{E}}$. The relation between $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{r}$ might be as shown in Fig. 17–5(b). As time goes on, the electric field rotates and the displacement rotates with the same frequency, so their relative orientation stays the same. Now let's look at the work being done on this electron. The rate that energy is being put into this electron is v, its velocity, times the component of $q\boldsymbol{\mathcal{E}}$ parallel to the velocity:

$$\frac{dW}{dt} = q\mathcal{E}_t v. \tag{17.31}$$

But look, there is angular momentum being poured into this electron, because there is always a torque about the origin. The torque is $q\mathcal{E}_t r$, which must be equal to the rate of change of angular momentum dJ_z/dt:

$$\frac{dJ_z}{dt} = q\mathcal{E}_t r. \tag{17.32}$$

Remembering that $v = \omega r$, we have that

$$\frac{dJ_z}{dW} = \frac{1}{\omega}.$$

Therefore, if we integrate the total angular momentum which is absorbed, it is proportional to the total energy—the constant of proportionality being $1/\omega$, which agrees with Eq. (17.30). Light does carry angular momentum—1 unit (times $\hbar$) if it is right circularly polarized along the z-axis, and -1 unit along the z-axis if it is left circularly polarized.

Now let's ask the following question: If light is linearly polarized in the x-direction, what is its angular momentum? Light polarized in the x-direction can be represented as the superposition of RHC and LHC polarized light. Therefore, there is a certain amplitude that the angular momentum is $+\hbar$ and another

† It is usually very convenient to measure angular momentum of atomic systems in units of $\hbar$. Then you can say that a spin one-half particle has angular momentum $\pm 1/2$ with respect to any axis. Or, in general, that the z-component of angular momentum is m. You don't need to repeat the $\hbar$ all the time.

amplitude that the angular momentum is $-\hbar$, so it doesn't have a *definite* angular momentum. It has an amplitude to appear with $+\hbar$ and an equal amplitude to appear with $-\hbar$. The interference of these two amplitudes produces the linear polarization, but it has *equal* probabilities to appear with plus or minus one unit of angular momentum. Macroscopic measurements made on a beam of linearly polarized light will show that it carries zero angular momentum, because in a large number of photons there are nearly equal numbers of RHC and LHC photons contributing opposite amounts of angular momentum—the average angular momentum is zero. And in the classical theory you don't find the angular momentum unless there is some circular polarization.

We have said that any spin-one particle can have three values of J_z, namely $+1, 0, -1$ (the three states we saw in the Stern-Gerlach experiment). But light is screwy; it has only two states. It does not have the zero case. This strange lack is related to the fact that light cannot stand still. For a particle of spin j which is standing still, there must be the $2j + 1$ possible states with values of j_z going in steps of 1 from $-j$ to $+j$. But it turns out that for something of spin j with zero mass only the states with the components $+j$ and $-j$ along the direction of motion exist. For example, light does not have three states, but only two—although a photon is still an object of spin one. How is this consistent with our earlier proofs—based on what happens under rotations in space—that for spin-one particles three states are necessary? For a particle at rest, rotations can be made about any axis without changing the momentum state. Particles with zero rest mass (like photons and neutrinos) cannot be at rest; only rotations about the axis along the direction of motion do not change the momentum state. Arguments about rotations around one axis only are insufficient to prove that three states are required, given that one of them varies as $e^{i\phi}$ under rotations by the angle ϕ.†

One further side remark. For a zero rest mass particle, in general, *only one* of the two spin states with respect to the line of motion $(+j, -j)$ is really necessary. For neutrinos—which are spin one-half particles—only the states with the component of angular momentum *opposite* to the direction of motion $(-\hbar/2)$ exist in nature [and only *along* the motion $(+\hbar/2)$ for antineutrinos]. When a system has inversion symmetry (so that parity is conserved, as it is for light) both components $(+j$, and $-j)$ are required.

17–5 The disintegration of the Λ^0

Now we want to give an example of how we use the theorem of conservation of angular momentum in a specifically quantum physical problem. We look at break-up of the lambda particle (Λ^0), which disintegrates into a proton and a π^- meson by a "weak" interaction:

$$\Lambda^0 \to p + \pi^-.$$

Assume we know that the pion has spin zero, that the proton has spin one-half, and that the Λ^0 has spin one-half. We would like to solve the following problem: Suppose that a Λ^0 were to be produced in a way that caused it to be completely polarized—by which we mean that its spin is, say "up," with respect to some suitably chosen z-axis—see Fig. 17–6(a). The question is, with what probability will it disintegrate so that the proton goes off at an angle θ with respect to the z-axis—as in Fig. 17–6(b)? In other words, what is the angular distribution of the disintegrations? We will look at the disintegration in the coordinate system in which the Λ^0 is at rest—we will measure the angles in this rest frame; then they can always be transformed to another frame if we want.

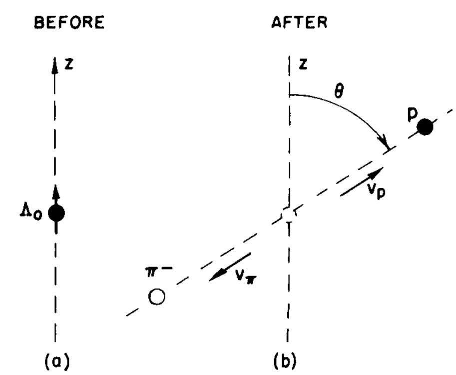

Fig. 17–6. A Λ^0 with spin "up" decays into a proton and a pion (in the CM system). What is the probability that the proton will go off at the angle θ?

† We have tried to find at least a proof that the component of angular momentum along the direction of motion must for a zero mass particle be an integral multiple of $\hbar/2$—and not something like $\hbar/3$. Even using all sorts of properties of the Lorentz transformation and what not, we failed. Maybe it's not true. We'll have to talk about it with Prof. Wigner, who knows all about such things.

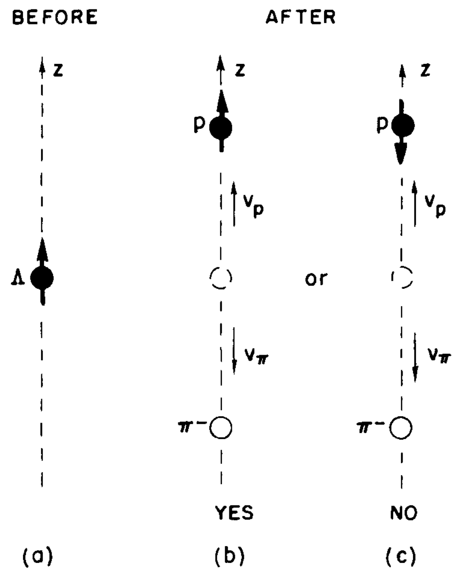

Fig. 17–7. Two possibilities for the decay of a spin "up" Λ^0 with the proton going along the $+z$-axis. Only (b) conserves angular momentum.

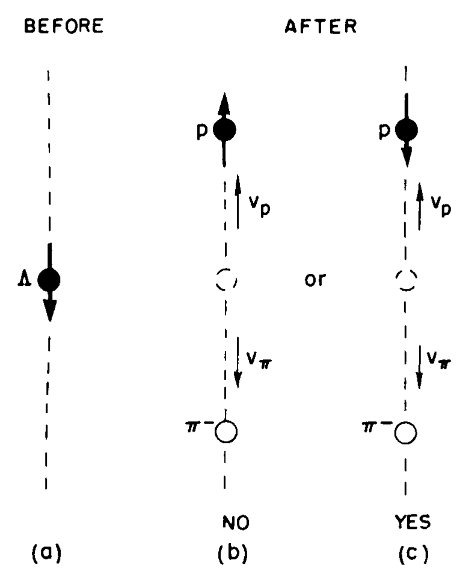

Fig. 17–8. The decay along the z-axis for a Λ^0 with spin "down."

We begin by looking at the special circumstance in which the proton is emitted into a small solid angle $\Delta\Omega$ along the z-axis (Fig. 17–7). Before the disintegration we have a Λ^0 with its spin "up," as in part (a) of the figure. After a short time—for reasons unknown to this day, except that they are connected with the weak decays—the Λ^0 explodes into a proton and a pion. Suppose the proton goes up along the $+z$-axis. Then, from the conservation of momentum, the pion must go down. Since the proton is a spin one-half particle, its spin must be either "up" or "down"—there are, in principle, the two possibilities shown in parts (b) and (c) of the figure. The conservation of angular momentum, however, requires that the proton have spin "up." This is most easily seen from the following argument. A particle moving along the z-axis cannot contribute any angular momentum about this axis by virtue of its motion; therefore, only the spins can contribute to J_z. The spin angular momentum about the z-axis is $+\hbar/2$ before the disintegration, so it must also be $+\hbar/2$ afterward. We can say that since the pion has no spin, the proton spin must be "up."

If you are worried that arguments of this kind may not be valid in quantum mechanics, we can take a moment to show you that they are. The initial state (before the disintegration), which we can call $|\,\Lambda^0, \text{spin } +z\rangle$ has the property that if it is rotated about the z-axis by the angle ϕ, the state vector gets multiplied by the phase factor $e^{i\phi/2}$. (In the rotated system the state vector is $e^{i\phi/2}\,|\,\Lambda^0, \text{spin } +z\rangle$.) That's what we mean by spin "up" for a spin one-half particle. Since nature's behavior doesn't depend on our choice of axes, the final state (the proton plus pion) must have the same property. We could write the final state as, say,

$$|\,\text{proton going } +z, \text{ spin } +z; \text{ pion going } -z\rangle.$$

But we really do not need to specify the pion motion, since in the frame we have chosen the pion always moves opposite the proton; we can simplify our description of the final state to

$$|\,\text{proton going } +z, \text{ spin } +z\rangle.$$

Now what happens to this state vector if we rotate the coordinates about the z-axis by the angle ϕ?

Since the proton and pion are moving along the z-axis, their motion isn't changed by the rotation. (That's why we picked this special case; we couldn't make the argument otherwise.) Also, nothing happens to the pion, because it is spin zero. The proton, however, has spin one-half. If its spin is "up" it will contribute a phase change of $e^{i\phi/2}$ in response to the rotation. (If its spin were "down" the phase change due to the proton would be $e^{-i\phi/2}$.) But the phase change with rotation before and after the excitement must be the same if angular momentum is to be conserved. (And it will be, since there are no outside influences in the Hamiltonian.) So the only possibility is that the proton spin will be "up." If the proton goes up, its spin must also be "up."

We conclude, then, that the conservation of angular momentum permits the process shown in part (b) of Fig. 17–7, but does not permit the process shown in part (c). Since we know that the disintegration occurs, there is some amplitude for process (b)—proton going up with spin "up." We'll let a stand for the amplitude that the disintegration occurs in this way in any infinitesimal interval of time.†

Now let's see what would happen if the Λ^0 spin were initially "down." Again we ask about the decays in which the proton goes up along the z-axis, as shown in Fig. 17–8. You will appreciate that in this case the proton must have spin "down" if angular momentum is conserved. Let's say that the amplitude for such a disintegration is b.

We can't say anything more about the two amplitudes a and b. They depend on the inner machinery of Λ^0, and the weak decays, and nobody yet knows how to

† We are now assuming that the machinery of the quantum mechanics is sufficiently familiar to you that we can speak about things in a physical way without taking the time to write down all the mathematical details. In case what we are saying here is not clear to you, we have put some of the missing details in a note at the end of the section.

calculate them. We'll have to get them from experiment. But with just these two amplitudes we *can* find out all we want to know about the angular distribution of the disintegration. We only have to be careful always to define completely the states we are talking about.

We want to know the probability that the proton will go off at the angle θ with respect to the z-axis (into a small solid angle $\Delta\Omega$) as drawn in Fig. 17–6. Let's put a new z-axis in this direction and call it the z'-axis. We know how to analyze what happens along this axis. With respect to this new axis, the Λ^0 no longer has its spin "up," but has a certain amplitude to have its spin "up" and another amplitude to have its spin "down." We have already worked these out in Chapter 6, and again in Chapter 10, Eq. (10.30). The amplitude to be spin "up" is $\cos\theta/2$, and the amplitude to be spin "down" is† $-\sin\theta/2$. When the Λ^0 spin is "up" along the z'-axis it will emit a proton in the $+z'$-direction with the amplitude a. So the amplitude to find an "up"-spinning proton coming out along the z'-direction is

$$a\cos\frac{\theta}{2}. \tag{17.33}$$

Similarly, the amplitude to find a "down"-spinning proton coming along the positive z'-axis is

$$-b\sin\frac{\theta}{2}. \tag{17.34}$$

The two processes that these amplitudes refer to are shown in Fig. 17–9.

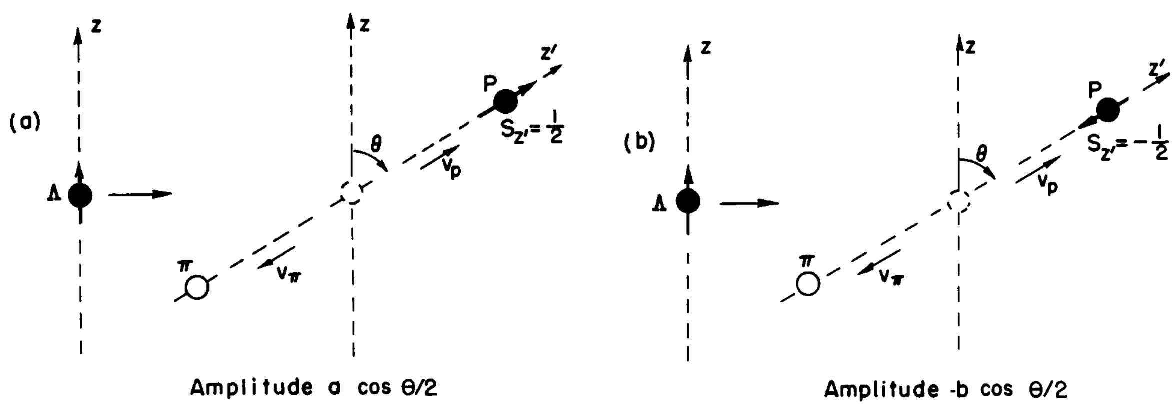

Fig. 17–9. Two possible decay states for the Λ^0.

Let's now ask the following easy question. If the Λ^0 has spin up along the z-axis, what is the probability that the decay proton will go off at the angle θ? The two spin states ("up" or "down" along z') are distinguishable even though we are not going to look at them. So to get the probability we square the amplitudes and add. The probability $f(\theta)$ of finding a proton in a small solid angle $\Delta\Omega$ at θ is

$$f(\theta) = |a|^2\cos^2\frac{\theta}{2} + |b|^2\sin^2\frac{\theta}{2}. \tag{17.35}$$

Remembering that $\sin^2\theta/2 = \frac{1}{2}(1-\cos\theta)$ and that $\cos^2\theta/2 = \frac{1}{2}(1+\cos\theta)$, we can write $f(\theta)$ as

$$f(\theta) = \left(\frac{|a|^2+|b|^2}{2}\right) + \left(\frac{|a|^2-|b|^2}{2}\right)\cos\theta. \tag{17.36}$$

† We have chosen to let z' be in the xz-plane and use the matrix elements for $R_y(\theta)$. You would get the same answer for any other choice.

The angular distribution has the form

$$f(\theta) = \beta(1 + \alpha \cos \theta). \tag{17.37}$$

The probability has one part that is independent of θ and one part that varies linearly with $\cos \theta$. From measuring the angular distribution we can get α and β, and therefore, $|a|$ and $|b|$.

Now there are many other questions we can answer. Are we interested only in protons with spin "up" along the *old* z-axis? Each of the terms in (17–33) and (17–34) will give an amplitude to find a proton with spin "up" and with spin "down" with respect to the z'-axis ($+z'$ and $-z'$). Spin "up" with respect to the old axis $|+z\rangle$ can be expressed in terms of the base states $|+z'\rangle$ and $|-z'\rangle$. We can then combine the two amplitudes (17.33) and (17.34) with the proper coefficients ($\cos \theta/2$ and $-\sin \theta/2$) to get the total amplitude

$$\left(a \cos^2 \frac{\theta}{2} + b \sin^2 \frac{\theta}{2}\right).$$

Its square is the probability that the proton comes out at the angle θ with its spin the same as the Λ^0 ("up" along the z-axis).

If parity were conserved, we could say one more thing. The disintegration of Fig. 17–8 is just the reflection—in say, the y z-plane of the disintegration of Fig. 17–7.† If parity were conserved, b would have to be equal to a or to $-a$. Then the coefficient α of (17.37) would be zero, and the disintegration would be equally likely to occur in all directions.

The experimental results show, however, that there *is* an asymmetry in the disintegration. The measured angular distribution does go as $\cos \theta$ as we predict—and not as $\cos^2 \theta$ or any other power. In fact, since the angular distribution has this form, we can deduce from these measurements that the spin of the Λ^0 is 1/2. Also, we see that parity is not conserved. In fact, the coefficient α is found experimentally to be -0.62 ± 0.05, so b is about twice as large as a. The lack of symmetry under a reflection is quite clear.

You see how much we can get from the conservation of angular momentum. We will give some more examples in the next chapter.

Parenthetical note. By the amplitude a in this section we mean the amplitude that the state $|\text{proton going } +z, \text{spin } +z\rangle$ is generated in an infinitesimal time dt from the state $|\Lambda, \text{spin } +z\rangle$, or, in other words, that

$$\langle \text{proton going } +z, \text{spin } +z \,|\, H \,|\, \Lambda, \text{spin } +z\rangle = i\hbar a, \tag{17.38}$$

where H is the Hamiltonian of the world—or, at least, of whatever is responsible for the Λ-decay. The conservation of angular momentum means that the Hamiltonian must have the property that

$$\langle \text{proton going } +z, \text{spin } -z \,|\, H \,|\, \Lambda, \text{spin } +z\rangle = 0. \tag{17.39}$$

By the amplitude b we mean that

$$\langle \text{proton going } +z, \text{spin } -z \,|\, H \,|\, \Lambda, \text{spin } -z\rangle = i\hbar b. \tag{17.40}$$

Conservation of angular momentum implies that

$$\langle \text{proton going } +z, \text{spin } +z \,|\, H \,|\, \Lambda, \text{spin } -z\rangle = 0. \tag{17.41}$$

If the amplitudes written in (17.33) and (17.34) are not clear, we can express them more mathematically as follows. By (17.33) we intend the amplitude that the Λ with spin along $+z$ will disintegrate into a proton moving along the $+z'$-direction with its spin also in the $+z'$-direction, namely the amplitude

$$\langle \text{proton going } +z', \text{spin } +z' \,|\, H \,|\, \Lambda, \text{spin } +z\rangle. \tag{17.42}$$

By the general theorems of quantum mechanics, this amplitude can be written as

$$\sum \langle \text{proton going } +z', \text{spin } +z' \,|\, H \,|\, \Lambda, i\rangle\langle \Lambda, i \,|\, \Lambda, \text{spin } +z\rangle, \tag{17.43}$$

† Remembering that the spin is an axial vector and flips over in the reflection.

where the sum is to be taken over the base states $| \Lambda, i\rangle$ of the Λ-particle at rest. Since the Λ-particle is spin one-half, there are two such base states which can be in any reference base we wish. If we use for base states spin "up" and spin "down" *with respect to z'* $(+z', -z')$, the amplitude of (17.43) is equal to the sum

$$\langle \text{proton going } +z', \text{ spin } +z' \mid H \mid \Lambda, +z'\rangle\langle\Lambda, +z' \mid \Lambda, +z\rangle$$
$$+\langle \text{proton going } +z', \text{ spin } +z' \mid H \mid \Lambda, -z'\rangle\langle\Lambda, -z' \mid \Lambda, +z\rangle. \tag{17.44}$$

The first factor of the first term is a, and the first factor of the second term is zero—from the definition of (17.38), and from (17.41), which in turn follows from angular momentum conservation. The remaining factor $\langle\Lambda, +z' \mid \Lambda, +z\rangle$ of the first term is just the amplitude that a spin one-half particle which has spin "up" along one axis will also have spin "up" along an axis tilted at the angle θ, which is $\cos\theta/2$—see Table 6-2. So (17.44) is just $a\cos\theta/2$, as we wrote in (17.33). The amplitude of (17.34) follows from the same kind of arguments for a spin "down" Λ-particle.

17–6 Summary of the rotation matrices

We would like now to bring together in one place the various things we have learned about the rotations for particles of spin one-half and spin one—so they will be convenient for future reference. On the next page you will find tables of the two rotation matrices $R_z(\phi)$ and $R_y(\theta)$ for spin one-half particles, for spin-one particles, and for photons (spin-one particles with zero rest mass). For each spin we will give the terms of the matrix $\langle j \mid R \mid i\rangle$ for rotations about the z-axis or the y-axis. They are, of course, exactly equivalent to the amplitudes like $\langle +T \mid 0\, S\rangle$ we have used in earlier chapters. We mean by $R_z(\phi)$ that the state is projected into a new coordinate system which is rotated through the angle ϕ about the z-axis—using always the right-hand rule to define the positive sense of the rotation. By $R_y(\theta)$ we mean that the reference axes are rotated by the angle θ about the y-axis. Knowing these two rotations, you can, of course, work out any arbitrary rotation. As usual, we write the matrix elements so that the state on the *left* is a base state of the *new* (rotated) frame and the state on the right is a base state of the old (unrotated) frame. You can interpret the entries in the tables in many ways. For instance, the entry $e^{-i\phi/2}$ in Table 17–1 means that the matrix element $\langle - \mid R \mid -\rangle = e^{-i\phi/2}$. It also means that $\hat{R} \mid -\rangle = e^{-i\phi/2} \mid -\rangle$, or that $\langle - \mid \hat{R} = \langle - \mid e^{-i\phi/2}$. It's all the same thing.

Table 17–1

Rotation matrices for spin one-half

Two states: $|+\rangle$, "up" along the z-axis, $m = +1/2$
$|-\rangle$, "down" along the z-axis, $m = -1/2$

$R_z(\phi)$	$\lvert+\rangle$	$\lvert-\rangle$
$\langle+\rvert$	$e^{+i\phi/2}$	0
$\langle-\rvert$	0	$e^{-i\phi/2}$

$R_y(\theta)$	$\lvert+\rangle$	$\lvert-\rangle$
$\langle+\rvert$	$\cos\theta/2$	$\sin\theta/2$
$\langle-\rvert$	$-\sin\theta/2$	$\cos\theta/2$

Table 17–2

Rotation matrices for spin one

Three states: $|+\rangle$, $m = +1$
$|0\rangle$, $m = 0$
$|-\rangle$, $m = -1$

$R_z(\phi)$	$\lvert+\rangle$	$\lvert0\rangle$	$\lvert-\rangle$
$\langle+\rvert$	$e^{+i\phi}$	0	0
$\langle0\rvert$	0	1	0
$\langle-\rvert$	0	0	$e^{-i\phi}$

$R_y(\theta)$	$\lvert+\rangle$	$\lvert0\rangle$	$\lvert-\rangle$
$\langle+\rvert$	$\frac{1}{2}(1+\cos\theta)$	$+\frac{1}{\sqrt{2}}\sin\theta$	$\frac{1}{2}(1-\cos\theta)$
$\langle0\rvert$	$-\frac{1}{\sqrt{2}}\sin\theta$	$\cos\theta$	$+\frac{1}{\sqrt{2}}\sin\theta$
$\langle-\rvert$	$\frac{1}{2}(1-\cos\theta)$	$-\frac{1}{\sqrt{2}}\sin\theta$	$\frac{1}{2}(1+\cos\theta)$

Table 17–3

Photons

Two states: $|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle)$, $m = +1$ (RHC polarized)

$|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle)$, $m = -1$ (LHC polarized)

$R_z(\phi)$	$\lvert R\rangle$	$\lvert L\rangle$
$\langle R\rvert$	$e^{+i\phi}$	0
$\langle L\rvert$	0	$e^{-i\phi}$