

## ***The Dependence of Amplitudes on Time***

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### **7-1 Atoms at rest; stationary states**

We want now to talk a little bit about the behavior of probability amplitudes in time. We say a “little bit,” because the actual behavior in time necessarily involves the behavior in space as well. Thus, we get immediately into the most complicated possible situation if we are to do it correctly and in detail. We are always in the difficulty that we can either treat something in a logically rigorous but quite abstract way, or we can do something which is not at all rigorous but which gives us some idea of a real situation—postponing until later a more careful treatment. With regard to energy dependence, we are going to take the second course. We will make a number of statements. We will not try to be rigorous—but will just be telling you things that have been found out, to give you some feeling for the behavior of amplitudes as a function of time. As we go along, the precision of the description will increase, so don’t get nervous that we seem to be picking things out of the air. It is, of course, all out of the air—the air of experiment and of the imagination of people. But it would take us too long to go over the historical development, so we have to plunge in somewhere. We could plunge into the abstract and deduce everything—which you would not understand—or we could go through a large number of experiments to justify each statement. We choose to do something in between.

An electron alone in empty space can, under certain circumstances, have a certain definite energy. For example, if it is standing still (so it has no translational motion, no momentum, or kinetic energy), it has its rest energy. A more complicated object like an atom can also have a definite energy when standing still, but it could also be internally excited to another energy level. (We will describe later the machinery of this.) We can often think of an atom in an excited state as having a definite energy, but this is really only approximately true. An atom doesn’t stay excited forever because it manages to discharge its energy by its interaction with the electromagnetic field. So there is some amplitude that a new state is generated—with the atom in a lower state, and the electromagnetic field in a higher state, of excitation. The total energy of the system is the same before and after, but the energy of the *atom* is reduced. So it is not precise to say an excited atom has a *definite* energy; but it will often be convenient and not too wrong to say that it does.

[Incidentally, why does it go one way instead of the other way? Why does an atom radiate light? The answer has to do with entropy. When the energy is in the electromagnetic field, there are so many different ways it can be—so many different places where it can wander—that if we look for the equilibrium condition, we find that in the most probable situation the field is excited with a photon, and the atom is de-excited. It takes a very long time for the photon to come back and find that it can knock the atom back up again. It’s quite analogous to the classical problem: Why does an accelerating charge radiate? It isn’t that it “wants” to lose energy, because, in fact, when it radiates, the energy of the world is the same as it was before. Radiation or absorption goes in the direction of increasing *entropy*.]

Nuclei can also exist in different energy levels, and in an approximation which disregards the electromagnetic effects, we can say that a nucleus in an excited state stays there. Although we know that it doesn’t stay there forever, it is often useful to start out with an approximation which is somewhat idealized and easier to think about. Also it is often a legitimate approximation under certain circumstances. (When we first introduced the classical laws of a falling body, we did not include friction, but there is almost never a case in which there isn’t *some* friction.)

### **7-1 Atoms at rest; stationary states**

### **7-2 Uniform motion**

### **7-3 Potential energy; energy conservation**

### **7-4 Forces; the classical limit**

### **7-5 The “precession” of a spin one-half particle**

*Review:* Chapter 17, Vol. I, *Space-Time*  
Chapter 48, Vol. I, *Beats*

Then there are the subnuclear “strange particles,” which have various masses. But the heavier ones disintegrate into other light particles, so again it is not correct to say that they have a precisely definite energy. That would be true only if they lasted forever. So when we make the approximation that they have a definite energy, we are forgetting the fact that they must blow up. For the moment, then, we will intentionally forget about such processes and learn later how to take them into account.

Suppose we have an atom—or an electron, or any particle—which at rest would have a definite energy  $E_0$ . By the energy  $E_0$  we mean the mass of the whole thing times  $c^2$ . This mass includes any internal energy; so an excited atom has a mass which is different from the mass of the same atom in the ground state. (The *ground state* means the state of lowest energy.) We will call  $E_0$  the “energy at rest.”

For an atom *at rest*, the quantum mechanical *amplitude* to find an atom at a place is the *same everywhere*; it does *not* depend on position. This means, of course, that the *probability* of *finding* the atom anywhere is the same. But it means even more. The *probability* could be independent of position, and still the *phase* of the *amplitude* could vary from point to point. But for a particle at rest, the complete amplitude is identical everywhere. It does, however, depend on the *time*. For a particle in a state of definite energy  $E_0$ , the amplitude to find the particle at  $(x, y, z)$  at the time  $t$  is

$$ae^{-i(E_0/\hbar)t}, \quad (7.1)$$

where  $a$  is some constant. The amplitude to be at any point in space is the same for all points, but depends on time according to (7.1). We shall simply assume this rule to be true.

Of course, we could also write (7.1) as

$$ae^{-i\omega t}, \quad (7.2)$$

with

$$\hbar\omega = E_0 = Mc^2,$$

where  $M$  is the rest mass of the atomic state, or particle. There are three different ways of specifying the energy: by the frequency of an amplitude, by the energy in the classical sense, or by the inertia. They are all equivalent; they are just different ways of saying the same thing.

You may be thinking that it is strange to think of a “particle” which has equal amplitudes to be found throughout all space. After all, we usually imagine a “particle” as a small object located “somewhere.” But don’t forget the uncertainty principle. If a particle has a definite energy, it has also a definite momentum. If the uncertainty in momentum is zero, the uncertainty relation,  $\Delta p \Delta x = \hbar$ , tells us that the uncertainty in the position must be infinite, and that is just what we are saying when we say that there is the same amplitude to find the particle at all points in space.

If the internal parts of an atom are in a different state with a different total energy, then the variation of the amplitude with time is different. If you don’t know in which state it is, there will be a certain amplitude to be in one state and a certain amplitude to be in another—and each of these amplitudes will have a different frequency. There will be an interference between these different components—like a beat-note—which can show up as a varying probability. Something will be “going on” inside of the atom—even though it is “at rest” in the sense that its center of mass is not drifting. However, if the atom has one definite energy, the amplitude is given by (7.1), and the absolute square of this amplitude does not depend on time. You see, then, that if a thing has a definite energy and if you ask any *probability* question about it, the answer is independent of time. Although the *amplitudes* vary with time, if the energy is *definite* they vary as an imaginary exponential, and the absolute value doesn’t change.

That’s why we often say that an atom in a definite energy level is in a *stationary state*. If you make any measurements of the things inside, you’ll find that nothing (in probability) will change in time. In order to have the probabilities change in

time, we have to have the interference of two amplitudes at two different frequencies, and that means that we cannot know what the energy is. The object will have one amplitude to be in a state of one energy and another amplitude to be in a state of another energy. That's the quantum mechanical description of something when its *behavior* depends on time.

If we have a "condition" which is a mixture of two different states with different energies, then the amplitude for each of the two states varies with time according to Eq. (7.2), for instance, as

$$e^{-i(E_1/\hbar)t} \quad \text{and} \quad e^{-i(E_2/\hbar)t}. \quad (7.3)$$

And if we have some combination of the two, we will have an interference. But notice that if we added a constant to both energies, it wouldn't make any difference. If somebody else were to use a different scale of energy in which all the energies were increased (or decreased) by a constant amount—say, by the amount  $A$ —then the amplitudes in the two states would, from his point of view, be

$$e^{-i(E_1+A)t/\hbar} \quad \text{and} \quad e^{-i(E_2+A)t/\hbar}. \quad (7.4)$$

All of his amplitudes would be multiplied by the same factor  $e^{-i(A/\hbar)t}$ , and all linear combinations, or interferences, would have the same factor. When we take the absolute squares to find the probabilities, all the answers would be the same. The choice of an origin for our energy scale makes no difference; we can measure the energy from any zero we want. For relativistic purposes it is nice to measure the energy so that the rest mass is included, but for many purposes that aren't relativistic it is often nice to subtract some standard amount from all energies that appear. For instance, in the case of an atom, it is usually convenient to subtract the energy  $M_s c^2$ , where  $M_s$  is the mass of all the *separate* pieces—the nucleus and the electrons—which is, of course, different from the mass of the atom. For other problems it may be useful to subtract from all energies the amount  $M_0 c^2$ , where  $M_0$  is the mass of the whole atom *in the ground state*; then the energy that appears is just the excitation energy of the atom. So, sometimes we may shift our zero of energy by some very large constant, but it doesn't make any difference, provided we shift all the energies in a particular calculation by the same constant. So much for a particle standing still.

## 7-2 Uniform motion

If we suppose that the relativity theory is right, a particle at rest in one inertial system can be in uniform motion in another inertial system. In the rest frame of the particle, the probability amplitude is the same for all  $x$ ,  $y$ , and  $z$  but varies with  $t$ . The *magnitude* of the amplitude is the same for all  $t$ , but the *phase* depends on  $t$ . We can get a kind of a picture of the behavior of the amplitude if we plot lines of equal phase—say, lines of zero phase—as a function of  $x$  and  $t$ . For a particle at rest, these equal-phase lines are parallel to the  $x$ -axis and are equally spaced in the  $t$ -coordinate, as shown by the dashed lines in Fig. 7-1.

In a different frame— $x'$ ,  $y'$ ,  $z'$ ,  $t'$ —that is moving with respect to the particle in, say, the  $x$ -direction, the  $x'$  and  $t'$  coordinates of any particular point in space are related to  $x$  and  $t$  by the Lorentz transformation. This transformation can be represented graphically by drawing  $x'$  and  $t'$  axes, as is done in Fig. 7-1. (See Chapter 17, Vol. I, Fig. 17-2.) You can see that in the  $x'$ - $t'$  system, points of equal phase† have a different spacing along the  $t'$ -axis, so the frequency of the time variation is different. Also there is a variation of the phase with  $x'$ , so the probability amplitude must be a function of  $x'$ .

† We are assuming that the phase should have the same value at corresponding points in the two systems. This is a subtle point, however, since the phase of a quantum mechanical amplitude is, to a large extent, arbitrary. A complete justification of this assumption requires a more detailed discussion involving interferences of two or more amplitudes.

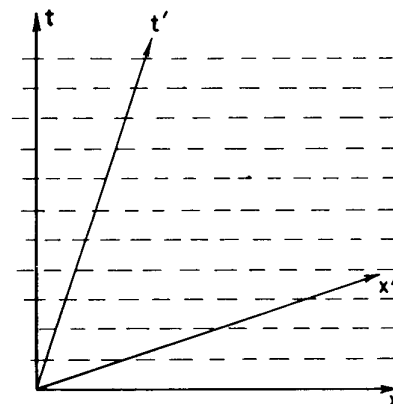


Fig. 7-1. Relativistic transformation of the amplitude of a particle at rest in the  $x$ - $t$  systems.

Under a Lorentz transformation for the velocity  $v$ , say along the negative  $x$ -direction, the time  $t$  is related to the time  $t'$  by

$$t = \frac{t' - x'v/c^2}{\sqrt{1 - v^2/c^2}},$$

so our amplitude now varies as

$$e^{-(i/\hbar)E_0 t} = e^{-(i/\hbar)(E_0 t' / \sqrt{1 - v^2/c^2} - E_0 v x' / c^2 \sqrt{1 - v^2/c^2})}$$

In the prime system it varies in space as well as in time. If we write the amplitude as

$$e^{-(i/\hbar)(E'_p t' - p' x')},$$

we see that  $E'_p = E_0 / \sqrt{1 - v^2/c^2}$  is the energy computed classically for a particle of rest energy  $E_0$  travelling at the velocity  $v$ , and  $p' = E'_p v / c^2$  is the corresponding particle momentum.

You know that  $x_\mu = (t, x, y, z)$  and  $p_\mu = (E, p_x, p_y, p_z)$  are four-vectors, and that  $p_\mu x_\mu = Et - \mathbf{p} \cdot \mathbf{x}$  is a scalar invariant. In the rest frame of the particle,  $p_\mu x_\mu$  is just  $Et$ ; so if we transform to another frame,  $Et$  will be replaced by

$$E't' - \mathbf{p}' \cdot \mathbf{x}'.$$

Thus, the probability amplitude of a particle which has the momentum  $\mathbf{p}$  will be proportional to

$$e^{-(i/\hbar)(E_p t - \mathbf{p} \cdot \mathbf{x})}, \quad (7.5)$$

where  $E_p$  is the energy of the particle whose momentum is  $\mathbf{p}$ , that is,

$$E_p = \sqrt{(pc)^2 + E_0^2}, \quad (7.6)$$

where  $E_0$  is, as before, the rest energy. For nonrelativistic problems, we can write

$$E_p = M_s c^2 + W_p, \quad (7.7)$$

where  $W_p$  is the energy over and above the rest energy  $M_s c^2$  of the parts of the atom. In general,  $W_p$  would include both the kinetic energy of the atom as well as its binding or excitation energy, which we can call the "internal" energy. We would write

$$W_p = W_{\text{int}} + \frac{p^2}{2M}, \quad (7.8)$$

and the amplitudes would be

$$e^{-(i/\hbar)(W_p t - \mathbf{p} \cdot \mathbf{x})}. \quad (7.9)$$

Because we will generally be doing nonrelativistic calculations, we will use this form for the probability amplitudes.

Note that our relativistic transformation has given us the variation of the amplitude of an atom which moves in space without any additional assumptions. The wave number of the space variations is, from (7.9),

$$k = \frac{p}{\hbar}; \quad (7.10)$$

so the wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{h}{p}. \quad (7.11)$$

This is the same wavelength we have used before for particles with the momentum  $p$ . This formula was first arrived at by de Broglie in just this way. For a moving particle, the *frequency* of the amplitude variations is still given by

$$\hbar\omega = W_p. \quad (7.12)$$

The absolute square of (7.9) is just 1, so for a particle in motion with a *definite energy*, the probability of finding it is the same everywhere and does not change with time. (It is important to notice that the amplitude is a *complex* wave. If we used a real sine wave, the square would vary from point to point, which would not be right.)

We know, of course, that there are situations in which particles move from place to place so that the probability depends on position and changes with time. How do we describe such situations? We can do that by considering amplitudes which are a superposition of two or more amplitudes for states of definite energy. We have already discussed this situation in Chapter 48 of Vol. I—even for probability amplitudes! We found that the sum of two amplitudes with different wave numbers  $k$  (that is, momenta) and frequencies  $\omega$  (that is, energies) gives interference humps, or beats, so that the square of the amplitude varies with space and time. We also found that these beats move with the so-called “group velocity” given by

$$v_g = \frac{\Delta\omega}{\Delta k},$$

where  $\Delta k$  and  $\Delta\omega$  are the differences between the wave numbers and frequencies for the two waves. For more complicated waves—made up of the sum of many amplitudes all near the same frequency—the group velocity is

$$v_g = \frac{d\omega}{dk}. \quad (7.13)$$

Taking  $\omega = E_p/\hbar$  and  $k = p/\hbar$ , we see that

$$v_g = \frac{dE_p}{dp}. \quad (7.14)$$

Using Eq. (7.6), we have

$$\frac{dE_p}{dp} = c^2 \frac{p}{E_p}. \quad (7.15)$$

But  $E_p = Mc^2$ , so

$$\frac{dE_p}{dp} = \frac{p}{M}, \quad (7.16)$$

which is just the classical velocity of the particle. Alternatively, if we use the non-relativistic expressions, we have

$$\omega = \frac{W_p}{\hbar} \quad \text{and} \quad k = \frac{p}{\hbar},$$

and

$$\frac{d\omega}{dk} = \frac{dW}{dp} = \frac{d}{dp} \left( \frac{p^2}{2M} \right) = \frac{p}{M}, \quad (7.17)$$

which is again the classical velocity.

Our result, then, is that if we have several amplitudes for pure energy states of nearly the same energy, their interference gives “lumps” in the probability that move through space with a velocity equal to the velocity of a classical particle of that energy. We should remark, however, that when we say we can add two amplitudes of different wave number together to get a beat-note that will correspond to a moving particle, we have introduced something new—something that we cannot deduce from the theory of relativity. We said what the amplitude did for a particle standing still and then deduced what it would do if the particle were moving. But we *cannot* deduce from these arguments what would happen when there are *two* waves moving with different speeds. If we stop one, we cannot stop the other. So we have added tacitly the *extra* hypothesis that not only is (7.9) a *possible* solution, but that there can also be solutions with all kinds of  $p$ 's for the same system, and that the different terms will interfere.

### 7-3 Potential energy; energy conservation

Now we would like to discuss what happens when the energy of a particle can change. We begin by thinking of a particle which moves in a force field described by a potential. We discuss first the effect of a constant potential. Suppose that we have a large metal can which we have raised to some electrostatic potential  $\phi$ , as in Fig. 7-2. If there are charged objects inside the can, their potential energy will be  $q\phi$ , which we will call  $V$ , and will be absolutely independent of position. Then there can be no change in the physics inside, because the constant potential doesn't make any difference so far as anything going on inside the can is concerned. Now there is no way we can deduce what the answer should be, so we must make a guess. The guess which works is more or less what you might expect: For the energy, we must use the sum of the potential energy  $V$  and the energy  $E_p$ —which is itself the sum of the internal and kinetic energies. The amplitude is proportional to

$$e^{-(i/\hbar)[(E_p+V)t - \mathbf{p}\cdot\mathbf{x}]} \quad (7.18)$$

The *general principle* is that the coefficient of  $t$ , which we may call  $\omega$ , is always given by the *total energy* of the system: internal (or "mass") energy, plus kinetic energy, plus potential energy:

$$\hbar\omega = E_p + V. \quad (7.19)$$

Or, for nonrelativistic situations,

$$\hbar\omega = W_{\text{int}} + \frac{p^2}{2M} + V. \quad (7.20)$$

Now what about physical phenomena inside the box? If there are several different energy states, what will we get? The amplitude for each state has the same additional factor

$$e^{-(i/\hbar)Vt}$$

over what it would have with  $V = 0$ . That is just like a change in the zero of our energy scale. It produces an equal phase change in all amplitudes, but as we have seen before, this doesn't change any of the probabilities. All the physical phenomena are the same. (We have assumed that we are talking about different states of the same charged object, so that  $q\phi$  is the same for all. If an object could change its charge in going from one state to another, we would have quite another result, but conservation of charge prevents this.)

So far, our assumption agrees with what we would expect for a change of energy reference level. But if it is really right, it should hold for a potential energy that is not just a constant. In general,  $V$  could vary in any arbitrary way with both time and space, and the complete result for the amplitude must be given in terms of a differential equation. We don't want to get concerned with the general case right now, but only want to get some idea about how some things happen, so we will think only of a potential that is constant in time and varies very slowly in space. Then we can make a comparison between the classical and quantum ideas.

Suppose we think of the situation in Fig. 7-3, which has two boxes held at the constant potentials  $\phi_1$  and  $\phi_2$  and a region in between where we will assume that the potential varies smoothly from one to the other. We imagine that some particle has an amplitude to be found in any one of the regions. We also assume that the momentum is large enough so that in any small region in which there are many wavelengths, the potential is nearly constant. We would then think that in any part of the space the amplitude ought to look like (7.18) with the appropriate  $V$  for that part of the space.

Let's think of a special case in which  $\phi_1 = 0$ , so that the potential energy there is zero, but in which  $q\phi_2$  is negative, so that classically the particle would have more energy in the second box. Classically, it would be going faster in the second box—it would have more energy and, therefore, more momentum. Let's see how that might come out of quantum mechanics.

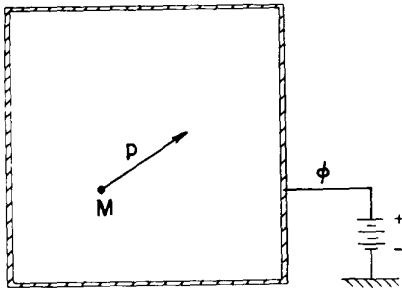


Fig. 7-2. A particle of mass  $M$  and momentum  $\mathbf{p}$  in a region of constant potential.

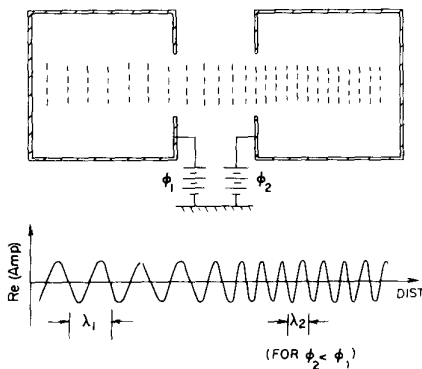


Fig. 7-3. The amplitude for a particle in transit from one potential to another.

With our assumption, the amplitude in the first box would be proportional to

$$e^{-(i/\hbar)[(W_{\text{int}}+p_1^2/2M+V_1)t-p_1\cdot x]}, \quad (7.21)$$

and the amplitude in the second box would be proportional to

$$e^{-(i/\hbar)[(W_{\text{int}}+p_2^2/2M+V_2)t-p_2\cdot x]}. \quad (7.22)$$

(Let's say that the internal energy is not being changed, but remains the same in both regions.) The question is: How do these two amplitudes match together through the region between the boxes?

We are going to suppose that the potentials are all constant in time—so that nothing in the conditions varies. We will then suppose that the variations of the amplitude (that is, its phase) have the same *frequency* everywhere—because, so to speak, there is nothing in the “medium” that depends on time. If nothing in the space is changing, we can consider that the wave in one region “generates” subsidiary waves all over space which will all oscillate at the same frequency—just as light waves going through materials at rest do not change their frequency. If the frequencies in (7.21) and (7.22) are the same, we must have that

$$W_{\text{int}} + \frac{p_1^2}{2M} + V_1 = W_{\text{int}} + \frac{p_2^2}{2M} + V_2. \quad (7.23)$$

Both sides are just the classical total energies, so Eq. (7.23) is a statement of the conservation of energy. In other words, the classical statement of the conservation of energy is equivalent to the quantum mechanical statement that the frequencies for a particle are everywhere the same if the conditions are not changing with time. It all fits with the idea that  $\hbar\omega = E$ .

In the special example that  $V_1 = 0$  and  $V_2$  is negative, Eq. (7.23) gives that  $p_2$  is greater than  $p_1$ , so the wavelength of the waves is shorter in region 2. The surfaces of equal phase are shown by the dashed lines in Fig. 7-3. We have also drawn a graph of the real part of the amplitude, which shows again how the wavelength decreases in going from region 1 to region 2. The group velocity of the waves, which is  $p/M$ , also increases in the way one would expect from the classical energy conservation, since it is just the same as Eq. (7.23).

There is an interesting special case where  $V_2$  gets so large that  $V_2 - V_1$  is greater than  $p_1^2/2M$ . Then  $p_2^2$ , which is given by

$$p_2^2 = 2M \left[ \frac{p_1^2}{2M} - V_2 + V_1 \right], \quad (7.24)$$

is *negative*. That means that  $p_2$  is an imaginary number, say,  $ip'$ . Classically, we would say that the particle never gets into region 2—it doesn't have enough energy to climb the potential hill. Quantum mechanically, however, the amplitude is still given by Eq. (7.22); its space variation still goes as

$$e^{(i/\hbar)p_2\cdot x}.$$

But if  $p_2$  is imaginary, the space dependence becomes a real exponential. Say that the particle was initially going in the  $+x$ -direction; then the amplitude would vary as

$$e^{-p'x/\hbar}. \quad (7.25)$$

The amplitude decreases rapidly with increasing  $x$ .

Imagine that the two regions at different potentials were very close together, so that the potential energy changed suddenly from  $V_1$  to  $V_2$ , as shown in Fig. 7-4(a). If we plot the real part of the probability amplitude, we get the dependence shown in part (b) of the figure. The wave in the first region corresponds to a particle trying to get into the second region, but the amplitude there falls off

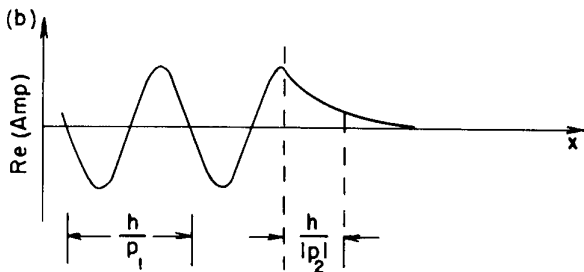
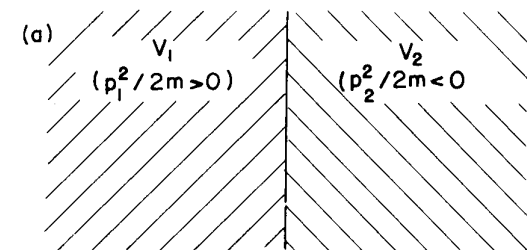


Fig. 7-4. The amplitude for a particle approaching a strongly repulsive potential.

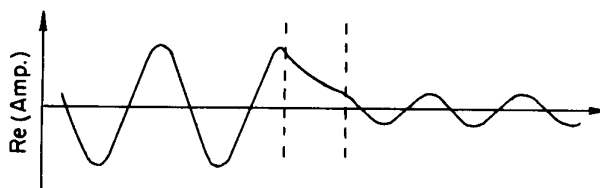
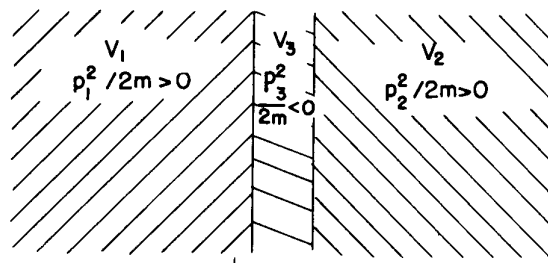


Fig. 7-5. The penetration of the amplitude through a potential barrier.

rapidly. There is some chance that it will be observed in the second region—where it could *never* get classically—but the amplitude is very small except right near the boundary. The situation is very much like what we found for the total internal reflection of light. The light doesn't normally get out, but we can observe it if we put something within a wavelength or two of the surface.

You will remember that if we put a second surface close to the boundary where light was totally reflected, we could get some light transmitted into the second piece of material. The corresponding thing happens to particles in quantum mechanics. If there is a narrow region with a potential  $V$ , so great that the classical kinetic energy would be negative, the particle would classically never get past. But quantum mechanically, the exponentially decaying amplitude can reach across the region and give a small probability that the particle will be found on the other side where the kinetic energy is again positive. The situation is illustrated in Fig. 7-5. This effect is called the quantum mechanical "penetration of a barrier."

The barrier penetration by a quantum mechanical amplitude gives the explanation—or description—of the  $\alpha$ -particle decay of a uranium nucleus. The potential energy of an  $\alpha$ -particle, as a function of the distance from the center, is shown in Fig. 7-6(a). If one tried to shoot an  $\alpha$ -particle with the energy  $E$  into the nucleus, it would feel an electrostatic repulsion from the nuclear charge  $z$  and would, classically, get no closer than the distance  $r_1$  where its total energy is equal to the potential energy  $V$ . Closer in, however, the potential energy is much lower because of the strong attraction of the short-range nuclear forces. How is it then that in radioactive decay we find  $\alpha$ -particles which started out inside the nucleus coming out with the energy  $E$ ? Because they start out with the energy  $E$  inside the nucleus and "leak" through the potential barrier. The probability amplitude is roughly as sketched in part (b) of Fig. 7-6, although actually the exponential decay is much larger than shown. It is, in fact, quite remarkable that the mean life of an  $\alpha$ -particle in the uranium nucleus is as long as  $4\frac{1}{2}$  billion years, when the natural oscillations inside the nucleus are so extremely rapid—about  $10^{22}$  per sec! How can one get a number like  $10^9$  years from  $10^{-22}$  sec? The answer is that the exponential gives the tremendously small factor of about  $e^{-45}$ —which gives the very small, though definite, probability of leakage. Once the  $\alpha$ -particle is in the nucleus, there is almost no amplitude at all for finding it outside; however, if you take many nuclei and wait long enough, you may be lucky and find one that has come out.

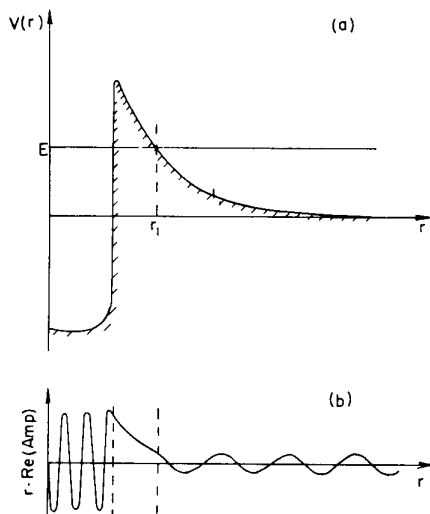


Fig. 7-6. (a) The potential function for an  $\alpha$ -particle in a uranium nucleus. (b) The qualitative form of the probability amplitude.



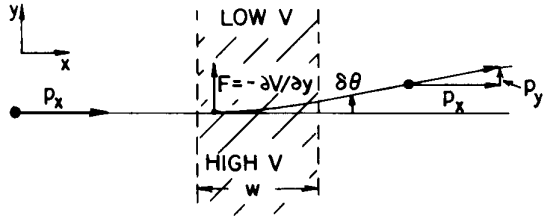


Fig. 7-7. The deflection of a particle by a transverse potential gradient.

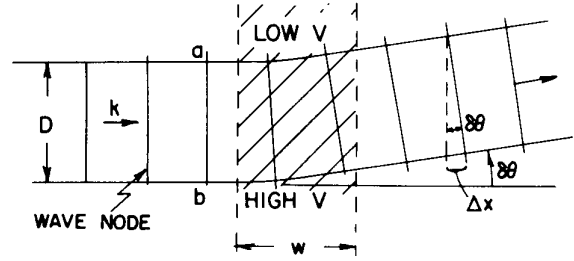


Fig. 7-8. The probability amplitude in a region with a transverse potential gradient.

#### 7-4 Forces; the classical limit

Suppose that we have a particle moving along and passing through a region where there is a potential that varies at right angles to the motion. Classically, we would describe the situation as sketched in Fig. 7-7. If the particle is moving along the  $x$ -direction and enters a region where there is a potential that varies with  $y$ , the particle will get a transverse acceleration from the force  $F = -\partial V/\partial y$ . If the force is present only in a limited region of width  $w$ , the force will act only for the time  $w/v$ . The particle will be given the transverse momentum

$$p_y = F \frac{w}{v}.$$

The angle of deflection  $\delta\theta$  is then

$$\delta\theta = \frac{p_y}{p} = \frac{Fw}{pv},$$

where  $p$  is the initial momentum. Using  $-\partial V/\partial y$  for  $F$ , we get

$$\delta\theta = -\frac{w}{pv} \frac{\partial V}{\partial y}. \quad (7.26)$$

It is now up to us to see if our idea that the waves go as (7.20) will explain the same result. We look at the same thing quantum mechanically, assuming that everything is on a very large scale compared with a wavelength of our probability amplitudes. In any small region we can say that the amplitude varies as

$$e^{-(i/\hbar)[(W+p^2/2M+V)t-p\cdot x]}. \quad (7.27)$$

Can we see that this will also give rise to a deflection of the particle when  $V$  has a transverse gradient? We have sketched in Fig. 7-8 what the waves of probability amplitude will look like. We have drawn a set of "wave nodes" which you can think of as surfaces where the phase of the amplitude is zero. In every small region, the wavelength—the distance between successive nodes—is

$$\lambda = \frac{h}{p},$$

where  $p$  is related to  $V$  through

$$W + \frac{p^2}{2M} + V = \text{const.} \quad (7.28)$$

In the region where  $V$  is larger,  $p$  is smaller, and the wavelength is longer. So the angle of the wave nodes gets changed as shown in the figure.

To find the change in angle of the wave nodes we notice that for the two paths  $a$  and  $b$  in Fig. 7-8 there is a difference of potential  $\Delta V = (\partial V/\partial y)D$ , so there is a difference  $\Delta p$  in the momentum along the two tracks which can be

obtained from (7.28):

$$\Delta \left( \frac{p^2}{2M} \right) = \frac{p}{M} \Delta p = -\Delta V. \quad (7.29)$$

The wave number  $p/\hbar$  is, therefore, different along the two paths, which means that the phase is advancing at a different rate. The difference in the rate of increase of phase is  $\Delta k = \Delta p/\hbar$ , so the accumulated phase difference in the total distance  $w$  is

$$\Delta(\text{phase}) = \Delta k \cdot w = \frac{\Delta p}{\hbar} \cdot w = -\frac{M}{p\hbar} \Delta V \cdot w. \quad (7.30)$$

This is the amount by which the phase on path  $b$  is “ahead” of the phase on path  $a$  as the wave leaves the strip. But outside the strip, a phase advance of this amount corresponds to the wave node being ahead by the amount

$$\Delta x = \frac{\lambda}{2\pi} \Delta(\text{phase}) = \frac{\hbar}{p} \Delta(\text{phase})$$

or

$$\Delta x = -\frac{M}{p^2} \Delta V \cdot w. \quad (7.31)$$

Referring to Fig. 7-8, we see that the new wavefronts will be at the angle  $\delta\theta$  given by

$$\Delta x = D \delta\theta; \quad (7.32)$$

so we have

$$D \delta\theta = -\frac{M}{p^2} \Delta V \cdot w. \quad (7.33)$$

This is identical to Eq. (7.26) if we replace  $p/m$  by  $v$  and  $\Delta V/D$  by  $\partial V/\partial y$ .

The result we have just got is correct only if the potential variations are slow and smooth—in what we call the *classical limit*. We have shown that under these conditions we will get the same particle motions we get from  $F = ma$ , provided we assume that a potential contributes a phase to the probability amplitude equal to  $Vt/\hbar$ . *In the classical limit, the quantum mechanics will agree with Newtonian mechanics.*

### 7-5 The “precession” of a spin one-half particle

Notice that we have not assumed anything special about the potential energy—it is just that energy whose derivative gives a force. For instance, in the Stern-Gerlach experiment we had the energy  $U = -\boldsymbol{\mu} \cdot \mathbf{B}$ , which gives a force if  $\mathbf{B}$  has a spatial variation. If we wanted to give a quantum mechanical description, we would have said that the particles in one beam had an energy that varied one way and that those in the other beam had an opposite energy variation. (We could put the magnetic energy  $U$  into the potential energy  $V$  or into the “internal” energy  $W$ ; it doesn’t matter.) Because of the energy variation, the waves are refracted, and the beams are bent up or down. (We see now that quantum mechanics would give us the same bending as we would compute from the classical mechanics.)

From the dependence of the amplitude on potential energy we would also expect that if a particle sits in a uniform magnetic field along the  $z$ -direction, its probability amplitude must be changing with time according to

$$e^{-(i/\hbar)(-\mu_z B)t}$$

(We can consider that this is, in effect, a definition of  $\mu_z$ .) In other words, if we place a particle in a uniform field  $B$  for a time  $\tau$ , its probability amplitude will be multiplied by

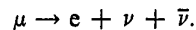
$$e^{-(i/\hbar)(-\mu_z B)\tau}$$

over what it would be in no field. Since for a spin one-half particle,  $\mu_z$  can be either plus or minus some number, say  $\mu$ , the two possible states in a uniform field would have their phases changing at the same rate but in opposite directions. The two amplitudes get multiplied by

$$e^{\pm(i/\hbar)\mu B\tau}. \quad (7.34)$$

This result has some interesting consequences. Suppose we have a spin one-half particle in some state that is not purely spin up or spin down. We can describe its condition in terms of the amplitudes to be in the pure up and pure down states. But in a magnetic field, these two states will have phases changing at a different rate. So if we ask some question about the amplitudes, the answer will depend on how long it has been in the field.

As an example, we consider the disintegration of the muon in a magnetic field. When muons are produced as disintegration products of  $\pi$ -mesons, they are polarized (in other words, they have a preferred spin direction). The muons, in turn, disintegrate—in about 2.2 microseconds on the average—emitting an electron and two neutrinos:



In this disintegration it turns out that (for at least the highest energies) the electrons are emitted preferentially in the direction opposite to the spin direction of the muon.

Suppose then that we consider the experimental arrangement shown in Fig. 7-9. If polarized muons enter from the left and are brought to rest in a block of material at  $A$ , they will, a little while later, disintegrate. The electrons emitted will, in general, go off in all possible directions. Suppose, however, that the muons all enter the stopping block at  $A$  with their spins in the  $x$ -direction. Without a magnetic field there would be some angular distribution of decay directions; we would like to know how this distribution is changed by the presence of the magnetic field. We expect that it may vary in some way with time. We can find out what happens by asking, for any moment, what the amplitude is that the muon will be found in the  $(+x)$  state.

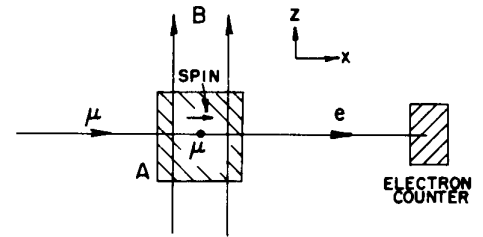


Fig. 7-9. A muon-decay experiment.

We can state the problem in the following way: A muon is known to have its spin in the  $+x$ -direction at  $t = 0$ ; what is the amplitude that it will be in the same state at the time  $\tau$ ? Now we do not have any rule for the behavior of a spin one-half particle in a magnetic field at right angles to the spin, but we do know what happens to the spin up and spin down states with respect to the field—their amplitudes get multiplied by the factor (7.34). Our procedure then is to choose the representation in which the base states are spin up and spin down with respect to the  $z$ -direction (the field direction). Any question can then be expressed with reference to the amplitudes for these states.

Let's say that  $\psi(t)$  represents the muon state. When it enters the block  $A$ , its state is  $\psi(0)$ , and we want to know  $\psi(\tau)$  at the later time  $\tau$ . If we represent the two base states by  $(+z)$  and  $(-z)$  we know the two amplitudes  $\langle +z | \psi(0) \rangle$  and  $\langle -z | \psi(0) \rangle$ —we know these amplitudes because we know that  $\psi(0)$  represents a state with the spin in the  $(+x)$  state. From the results of the last chapter, these amplitudes are†

$$\langle +z | +x \rangle = C_+ = \frac{1}{\sqrt{2}} \quad (7.35)$$

and

$$\langle -z | +x \rangle = C_- = \frac{1}{\sqrt{2}}.$$

They happen to be equal. Since these amplitudes refer to the condition at  $t = 0$ , let's call them  $C_+(0)$  and  $C_-(0)$ .

† If you skipped Chapter 6, you can just take (7.35) as an underived rule for now. We will give later (in Chapter 10) a more complete discussion of spin precession, including a derivation of these amplitudes.

Now we know what happens to these two amplitudes with time. Using (7.34), we have

$$C_+(t) = C_+(0)e^{-(i/\hbar)\mu B t}$$

and

$$C_-(t) = C_-(0)e^{+(i/\hbar)\mu B t}$$

But if we know  $C_+(t)$  and  $C_-(t)$ , we have all there is to know about the condition at  $t$ . The only trouble is that what we *want* to know is the probability that at  $t$  the spin will be in the  $+x$ -direction. Our general rules can, however, take care of this problem. We write that the amplitude to be in the  $(+x)$  state at time  $t$ , which we may call  $A_+(t)$ , is

$$A_+(t) = \langle +x | \psi(t) \rangle = \langle +x | +z \rangle \langle +z | \psi(t) \rangle + \langle +x | -z \rangle \langle -z | \psi(t) \rangle$$

or

$$A_+(t) = \langle +x | +z \rangle C_+(t) + \langle +x | -z \rangle C_-(t). \quad (7.37)$$

Again using the results of the last chapter—or better the equality  $\langle \phi | \chi \rangle = \langle \chi | \phi \rangle^*$  from Chapter 5—we know that

$$\langle +x | +z \rangle = \frac{1}{\sqrt{2}} \quad \langle +x | -z \rangle = \frac{1}{\sqrt{2}}.$$

So we know all the quantities in Eq. (7.37). We get

$$A_+(t) = \frac{1}{2}e^{(i/\hbar)\mu B t} + \frac{1}{2}e^{-(i/\hbar)\mu B t}.$$

or

$$A_+(t) = \cos \frac{\mu B}{\hbar} t.$$

A particularly simple result! Notice that the answer agrees with what we expect for  $t = 0$ . We get  $A_+(0) = 1$ , which is right, because we assumed that the muon was in the  $(+x)$  state at  $t = 0$ .

The probability  $P_+$  that the muon will be found in the  $(+x)$  state at  $t$  is  $(A_+)^2$  or

$$P_+ = \cos^2 \frac{\mu B t}{\hbar}.$$

The probability oscillates between zero and one, as shown in Fig. 7-10. Note that the probability returns to one for  $\mu B t / \hbar = \pi$  (not  $2\pi$ ). Because we have squared the cosine function, the probability repeats itself with the frequency  $2\mu B / \hbar$ .

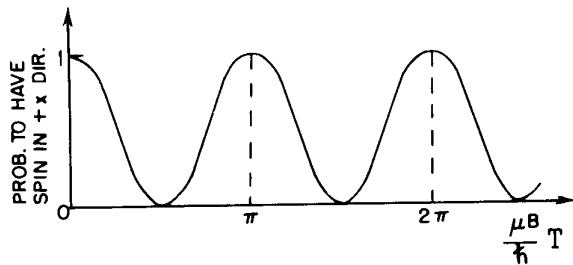


Fig. 7-10. Time dependence of the probability that a spin one-half particle will be in a  $(+)$  state with respect to the  $x$ -axis.

Thus, we find that the chance of catching the decay electron in the electron counter of Fig. 7-9 varies periodically with the length of time the muon has been sitting in the magnetic field. The frequency depends on the magnetic moment  $\mu$ . The magnetic moment of the muon has, in fact, been measured in just this way.

We can, of course, use the same method to answer any other questions about the muon decay. For example, how does the chance of detecting a decay electron

in the  $y$ -direction at  $90^\circ$  to the  $x$ -direction but still at right angles to the field depend on  $t$ ? If you work it out, the amplitude to be in the  $(+y)$  state varies as  $\cos^2\{(\mu B t/\hbar) - \pi/4\}$ , which oscillates with the same period but reaches its maximum one-quarter cycle later, when  $\mu B t/\hbar = \pi/4$ . In fact, what is happening is that as time goes on, the muon goes through a succession of states which correspond to complete polarization in a direction that is continually rotating about the  $z$ -axis. We can describe this by saying that *the spin is precessing* at the frequency

$$\omega_p = \frac{2\mu B}{\hbar}. \quad (7.38)$$

You can begin to see the form that our quantum mechanical description will take when we are describing how things behave in time.