31

Tensors

Review: Chapter 11, Vol I, *Vectors*
Chapter 20, Vol. I, *Rotation in Space*

31–1 The tensor of polarizability

Physicists always have a habit of taking the simplest example of any phenomenon and calling it "physics," leaving the more complicated examples to become the concern of other fields—say of applied mathematics, electrical engineering, chemistry, or crystallography. Even solid-state physics is almost only half physics because it worries too much about special substances. So in these lectures we will be leaving out many interesting things. For instance, one of the important properties of crystals—or of most substances—is that their electric polarizability is different in different directions. If you apply a field in any direction, the atomic charges shift a little and produce a dipole moment, but the magnitude of the moment depends very much on the direction of the field. That is, of course, quite a complication. But in physics we usually start out by talking about the special case in which the polarizability is the same in all directions, to make life easier. We leave the other cases to some other field. Therefore, for our later work, we will not need at all what we are going to talk about in this chapter.

The mathematics of tensors is particularly useful for describing properties of substances which vary in direction—although that's only one example of their use. Since most of you are not going to become physicists, but are going to go into the *real* world, where things depend severely upon direction, sooner or later you will need to use tensors. In order not to leave anything out, we are going to describe tensors, although not in great detail. We want the feeling that our treatment of physics is complete. For example, our electrodynamics is complete—as complete as any electricity and magnetism course, even a graduate course. Our mechanics is not complete, because we studied mechanics when you didn't have a high level of mathematical sophistication, and we were not able to discuss subjects like the principle of least action, or Lagrangians, or Hamiltonians, and so on, which are *more elegant ways* of describing mechanics. Except for general relativity, however, we do have the complete *laws* of mechanics. Our electricity and magnetism is complete, and a lot of other things are quite complete. The quantum mechanics, naturally, will not be—we have to leave something for the future. But you should at least know what a tensor is.

We emphasized in Chapter 30 that the properties of crystalline substances are different in different directions—we say they are *anisotropic*. The variation of the induced dipole moment with the direction of the applied electric field is only one example, the one we will use for our example of a tensor. Let's say that for a given direction of the electric field the induced dipole moment per unit volume $\boldsymbol{P}$ is proportional to the strength of the applied field $\boldsymbol{E}$. (This is a good approximation for many substances if $\boldsymbol{E}$ is not too large.) We will call the proportionality constant α.* We want now to consider substances in which α depends on the direction of the applied field, as, for example, in crystals like calcite, which make double images when you look through them.

Suppose, in a particular crystal, we find that an electric field $\boldsymbol{E}_1$ in the x-direction produces the polarization $\boldsymbol{P}_1$ in the x-direction. Then we find that an electric field $\boldsymbol{E}_2$ in the y-direction, with the same *strength*, as $\boldsymbol{E}_1$ produces a different polar-

* In Chapter 10 we followed the usual convention and wrote $P = \epsilon_0 \chi E$ and called χ ("khi") the "susceptibility." Here, it will be more convenient to use a single letter, so we write α for $\epsilon_0\chi$. For isotropic dielectrics, $\alpha = (\kappa - 1)\epsilon_0$, where κ is the dielectric constant (see Section 10–4).

ization $\boldsymbol{P}_2$ in the y-direction. What would happen if we put an electric field at 45°? Well, that's a superposition of two fields along x and y, so the polarization $\boldsymbol{P}$ will be the vector sum of $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, as shown in Fig. 31–1(a). The polarization is no longer in the same direction as the electric field. You can see how that might come about. There may be charges which can move easily up and down, but which are rather stiff for sidewise motions. When a force is applied at 45°, the charges move farther up than they do toward the side. The displacements are not in the direction of the external force, because there are asymmetric internal elastic forces.

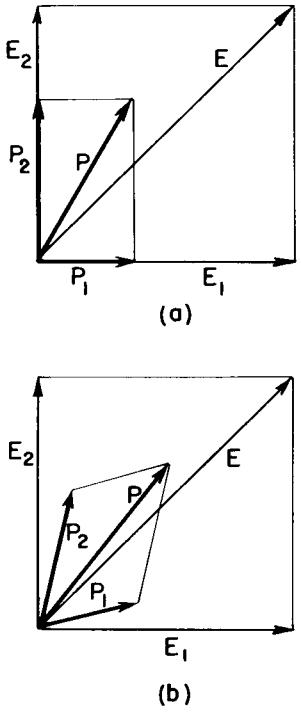

Fig. 31–1. The vector addition of polarizations in an anisotropic crystal.

There is, of course, nothing special about 45°. It is *generally* true that the induced polarization of a crystal is *not* in the direction of the electric field. In our example above, we happened to make a "lucky" choice of our x- and y-axes, for which $\boldsymbol{P}$ was along $\boldsymbol{E}$ for both the x- and y-directions. If the crystal were rotated with respect to the coordinate axes, the electric field $\boldsymbol{E}_2$ in the y-direction would have produced a polarization $\boldsymbol{P}$ with both an x- and a y-component. Similarly, the polarization due to an electric field in the x-direction would have produced a polarization with an x-component and a y-component. Then the polarizations would be as shown in Fig. 31–1(b), instead of as in part (a). Things get more complicated—but for any field $\boldsymbol{E}$, the *magnitude* of $\boldsymbol{P}$ is still proportional to the magnitude of $\boldsymbol{E}$.

We want now to treat the general case of an arbitrary orientation of a crystal with respect to the coordinate axes. An electric field in the x-direction will produce a polarization $\boldsymbol{P}$ with x-, y-, and z-components; we can write

$$P_x = \alpha_{xx}E_x, \qquad P_y = \alpha_{yx}E_x, \qquad P_z = \alpha_{zx}E_x. \tag{31.1}$$

All we are saying here is that if the electric field is in the x-direction, the polarization does not have to be in that same direction, but rather has an x-, a y-, and a z-component—each proportional to E_x. We are calling the constants of proportionality α_{xx}, α_{yx}, and α_{zx}, respectively (the first letter to tell us which component of P is involved, the last to refer to the direction of the electric field).

Similarly, for a field in the y-direction, we can write

$$P_x = \alpha_{xy}E_y, \qquad P_y = \alpha_{yy}E_y, \qquad P_z = \alpha_{zy}E_y; \tag{31.2}$$

and for a field in the z-direction,

$$P_x = \alpha_{xz}E_z, \qquad P_y = \alpha_{yz}E_z, \qquad P_z = \alpha_{zz}E_z. \tag{31.3}$$

Now we have said that polarization depends linearly on the fields, so if there is an electric field $\boldsymbol{E}$ that has both an x- and a y-component, the resulting x-component of $\boldsymbol{P}$ will be the sum of the two P_x's of Eqs. (31.1) and (31.2). If $\boldsymbol{E}$ has components along x, y, and z, the resulting components of $\boldsymbol{P}$ will be the sum of the three contributions in Eqs. (31.1), (31.2), and (31.3). In other words, $\boldsymbol{P}$ will be given by

$$\begin{aligned} P_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z, \\ P_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z, \\ P_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z. \end{aligned} \tag{31.4}$$

The dielectric behavior of the crystal is then completely described by the nine quantities (α_{xx}, α_{xy}, α_{xz}, α_{yz}, . . .), which we can represent by the symbol α_{ij}. (The subscripts i and j each stand for any one of the three possible letters x, y, and z.) Any arbitrary electric field $\boldsymbol{E}$ can be resolved with the components E_x, E_y, and E_z; from these we can use the α_{ij} to find P_x, P_y, and P_z, which together give the total polarization $\boldsymbol{P}$. The set of nine coefficients α_{ij} is called a *tensor*—in this instance, the *tensor of polarizability*. Just as we say that the three numbers (E_x, E_y, E_z) "form the vector $\boldsymbol{E}$," we say that the nine numbers (α_{xx}, α_{xy}, . . .) "form the tensor α_{ij}."

31–2 Transforming the tensor components

You know that when we change to a different coordinate system x', y', and z', the components $E_{x'}$, $E_{y'}$, and $E_{z'}$ of the vector will be quite different—as will also *the components* of $\boldsymbol{P}$. So all the coefficients α_{ij} will be different for a different set of coordinates. You can, in fact, see how the α's must be changed by changing the components of $\boldsymbol{E}$ and $\boldsymbol{P}$ in the proper way, because if we describe the *same physical* electric field in the new coordinate system we should get the same polarization. For any new set of coordinates, $P_{x'}$ is a linear combination of P_x, P_y, and P_z:

$$P_{x'} = aP_x + bP_y + cP_z,$$

and similarly for the other components. If you substitute for P_x, P_y, and P_z in terms of the E's, using Eq. (31.4), you get

$$\begin{aligned} P_{x'} = {} & a(\alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z) \\ & + b(\alpha_{yx}E_x + \alpha_{yy}E_y + \cdots) \\ & + c(\alpha_{zx}E_x + \quad \cdots \quad + \cdots). \end{aligned}$$

Then you write E_x, E_y, and E_z in terms of $E_{x'}$, $E_{y'}$, and $E_{z'}$; for instance,

$$E_x = a'E_{x'} + b'E_{y'} + c'E_{z'},$$

where a', b', c' are related to, but not equal to, a, b, c. So you have $P_{x'}$, expressed in terms of the components $E_{x'}$, $E_{y'}$, and $E_{z'}$; that is, you have the new α_{ij}. It is fairly messy, but quite straightforward.

When we talk about changing the axes we are assuming that the crystal stays put *in space*. If the crystal were rotated *with* the axes, the α's would not change. Conversely, if the orientation of the crystal were changed with respect to the axes, we would have a new set of α's. But if they are known for *any* one orientation of the crystal, they can be found for any other orientation by the transformation we have just described. In other words, the dielectric property of a crystal is described *completely* by giving the components of the polarization tensor α_{ij} with respect to any arbitrarily chosen set of axes. Just as we can associate a vector velocity $\boldsymbol{v} = (v_x, v_y, v_z)$ with a particle, knowing that the three components will change in a certain definite way if we change our coordinate axes, so with a crystal we associate its polarization tensor α_{ij}, whose nine components will transform in a certain definite way if the coordinate system is changed.

The relation between $\boldsymbol{P}$ and $\boldsymbol{E}$ written in Eq. (31.4) can be put in the more compact notation:

$$P_i = \sum_j \alpha_{ij}E_j, \tag{31.5}$$

where it is understood that i represents either x, y, or z and that the sum is taken on $j = x$, y, and z. Many special notations have been invented for dealing with tensors, but each of them is convenient only for a limited class of problems. One common convention is to omit the sum sign ($\sum$) in Eq. (31.5), leaving it *understood* that whenever the same subscript occurs twice (here j), a sum is to be taken over that index. Since we will be using tensors so little, we will not bother to adopt any such special notations or conventions.

31–3 The energy ellipsoid

We want now to get some experience with tensors. Suppose we ask the interesting question: What energy is required to polarize the crystal (in addition to the energy in the electric field which we know is $\epsilon_0E^2/2$ per unit volume)? Consider for a moment the atomic charges that are being displaced. The work done in displacing the charge the distance dx is $qE_x\,dx$, and if there are N charges per unit volume, the work done is $qE_xN\,dx$. But $qN\,dx$ is the change dP_x in the dipole

moment per unit volume. So the energy required *per unit volume* is

$$E_x\, dP_x.$$

Combining the work for the three components of the field, the work per unit volume is found to be

$$\boldsymbol{E} \cdot d\boldsymbol{P}.$$

Since the magnitude of $\boldsymbol{P}$ is proportional to $\boldsymbol{E}$, the work done per unit volume in bringing the polarization from 0 to $\boldsymbol{P}$ is the integral of $\boldsymbol{E} \cdot d\boldsymbol{P}$. Calling this work u_P,* we write

$$u_P = \tfrac{1}{2}\boldsymbol{E} \cdot \boldsymbol{P} = \tfrac{1}{2}\sum_i E_i P_i. \tag{31.6}$$

Now we can express $\boldsymbol{P}$ in terms of $\boldsymbol{E}$ by Eq. (31.5), and we have that

$$u_P = \tfrac{1}{2}\sum_i \sum_j \alpha_{ij} E_i E_j. \tag{31.7}$$

The energy density u_P is a number independent of the choice of axes, so it is a scalar. A tensor has then the property that when it is summed over one index (with a vector), it gives a new vector; and when it is summed over *both* indexes (with *two* vectors), it gives a scalar.

The tensor α_{ij} should really be called a "tensor of second rank," because it has two indexes. A vector—with *one* index—is a tensor of the first rank, and a scalar—with no index—is a tensor of zero rank. So we say that the electric field $\boldsymbol{E}$ is a tensor of the first rank and that the energy density u_P is a tensor of zero rank. It is possible to extend the ideas of a tensor to three or more indexes, and so to make tensors of ranks higher than two.

The subscripts of the polarization tensor range over three possible values—they are tensors in three dimensions. The mathematicians consider tensors in four, five, or more dimensions. We have already used a four-dimensional tensor $F_{\mu\nu}$ in our relativistic description of the electromagnetic field (Chapter 26).

The polarization tensor α_{ij} has the interesting property that it is *symmetric*, that is, that $\alpha_{xy} = \alpha_{yx}$, and so on for any pair of indexes. (This is a *physical* property of a real crystal and not necessary for all tensors.) You can prove for yourself that this must be true by computing the change in energy of a crystal through the following cycle: (1) Turn on a field in the x-direction; (2) turn on a field in the y-direction; (3) turn *off* the x-field; (4) turn off the y-field. The crystal is now back where it started, and the net work done on the polarization must be back to zero. You can show, however, that for this to be true, α_{xy} must be equal to α_{yx}. The same kind of argument can, of course, be given for α_{xz}, etc. So the polarization tensor is symmetric.

This also means that the polarization tensor can be measured by just measuring the energy required to polarize the crystal in various directions. Suppose we apply an $\boldsymbol{E}$-field with only an x- and a y-component; then according to Eq. (31.7),

$$u_P = \tfrac{1}{2}[\alpha_{xx}E_x^2 + (\alpha_{xy} + \alpha_{yx})E_x E_y + \alpha_{yy}E_y^2]. \tag{31.8}$$

With an E_x alone, we can determine α_{xx}; with an E_y alone, we can determine α_{yy}; with both E_x and E_y, we get an extra energy due to the term with $(\alpha_{xy} + \alpha_{yx})$. Since the α_{xy} and α_{yx} are equal, this term is $2\alpha_{xy}$ and can be related to the energy.

The energy expression, Eq. (31.8), has a nice geometric interpretation. Suppose we ask what fields E_x and E_y correspond to some *given* energy density—say u_0. That is just the mathematical problem of solving the equation

$$\alpha_{xx}E_x^2 + 2\alpha_{xy}E_x E_y + \alpha_{yy}E_y^2 = 2u_0.$$

This is a quadratic equation, so if we plot E_x and E_y, the solutions of this equation

* This work done in *producing* the polarization by an electric field is not to be confused with the potential energy $-\boldsymbol{p}_0 \cdot \boldsymbol{E}$ of a permanent dipole moment p_0.

are all the points on an ellipse (Fig. 31–2). (It must be an ellipse, rather than a parabola or a hyperbola, because the energy for any field is always positive and finite.) The vector $\boldsymbol{E}$ with components E_x and E_y can be drawn from the origin to the ellipse. So such an "energy ellipse" is a nice way of "visualizing" the polarization tensor.

If we now generalize to include all three components, the electric vector $\boldsymbol{E}$ in *any* direction required to give a unit energy density gives a point which will be on the surface of an ellipsoid, as shown in Fig. 31–3. The shape of this ellipsoid of constant energy uniquely characterizes the tensor polarizability.

Now an ellipsoid has the nice property that it can always be described simply by giving the directions of three "principal axes" and the diameters of the ellipse along these axes. The "principal axes" are the directions of the longest and shortest diameters and the direction at right angles to both. They are indicated by the axes a, b, and c in Fig. 31–3. With respect to these axes, the ellipsoid has the particularly simple equation

$$\alpha_{aa}E_a^2 + \alpha_{bb}E_b^2 + \alpha_{cc}E_c^2 = 2u_0.$$

So with respect to these axes, the dielectric tensor has only three components that are not zero: α_{aa}, α_{bb}, and α_{cc}. That is to say, no matter how complicated a crystal is, it is always possible to choose a set of axes (not necessarily the crystal axes) for which the polarization tensor has only three components. With such a set of axes, Eq. (31.4) becomes simply

$$P_a = \alpha_{aa}E_a, \qquad P_b = \alpha_{bb}E_b, \qquad P_c = \alpha_{cc}E_c. \tag{31.9}$$

An electric field along any one of the principal axes produces a polarization along the same axis, but the coefficients for the three axes may, of course, be different.

Often, a tensor is described by listing the nine coefficients in a table inside of a pair of brackets:

$$\begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix}. \tag{31.10}$$

For the principal axes a, b, and c, only the diagonal terms are not zero; we say then that "the tensor is diagonal." The complete tensor is

$$\begin{bmatrix} \alpha_{aa} & 0 & 0 \\ 0 & \alpha_{bb} & 0 \\ 0 & 0 & \alpha_{cc} \end{bmatrix}. \tag{31.11}$$

The important point is that any polarization tensor (in fact, *any symmetric* tensor of rank two in any number of dimensions) can be put in this form by choosing a suitable set of coordinate axes.

If the three elements of the polarization tensor in diagonal form are all equal, that is, if

$$\alpha_{aa} = \alpha_{bb} = \alpha_{cc} = \alpha, \tag{31.12}$$

the energy ellipsoid becomes a sphere, and the polarizability is the same in all directions. The material is isotropic. In the tensor notation,

$$\alpha_{ij} = \alpha\delta_{ij}, \tag{31.13}$$

where δ_{ij} is the *unit tensor*

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{31.14}$$

That means, of course,

$$\begin{aligned} \delta_{ij} &= 1, \quad \text{if} \quad i = j; \\ \delta_{ij} &= 0, \quad \text{if} \quad i \neq j. \end{aligned} \tag{31.15}$$

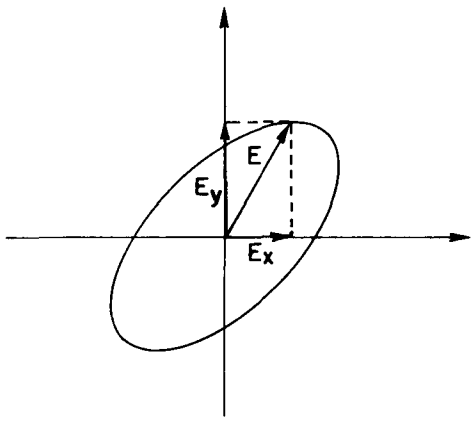

Fig. 31–2. Locus of the vector $\boldsymbol{E} = (E_x, E_y)$ that gives a constant energy of polarization.

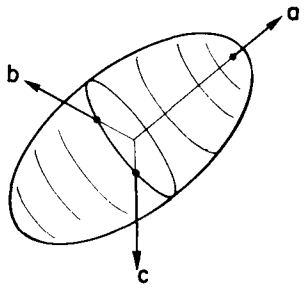

Fig. 31–3. The energy ellipsoid of the polarization tensor.

The tensor δ_{ij} is often called the "Kronecker delta." You may amuse yourself by proving that the tensor (31.14) has exactly the same form if you change the coordinate system to any other rectangular one. The polarization tensor of Eq. (31.13) gives

$$P_i = \alpha \sum_j \delta_{ij} E_j = \alpha E_i,$$

which means the same as our old result for isotropic dielectrics:

$$\boldsymbol{P} = \alpha \boldsymbol{E}.$$

The shape and orientation of the polarization ellipsoid can sometimes be related to the symmetry properties of the crystal. We have said in Chapter 30 that there are 230 different possible internal symmetries of a three-dimensional lattice and that they can, for many purposes, be conveniently grouped into seven classes, according to the shape of the unit cell. Now the ellipsoid of polarizability must share the internal geometric symmetries of the crystal. For example, a triclinic crystal has low symmetry—the ellipsoid of polarizability will have unequal axes, and its orientation will not, in general, be aligned with the crystal axes. On the other hand, a monoclinic crystal has the property that its properties are unchanged if the crystal is rotated 180° about one axis. So the polarization tensor must be the same after such a rotation. It follows that the ellipsoid of the polarizability must return to itself after a 180° rotation. That can happen only if one of the axes of the ellipsoid is in the same direction as the symmetry axis of the crystal. Otherwise, the orientation and dimensions of the ellipsoid are unrestricted

For an orthorhombic crystal, however, the axes of the ellipsoid must correspond to the crystal axes, because a 180° rotation about any one of the three axes repeats the same lattice. If we go to a tetragonal crystal, the ellipse must have the same symmetry, so it must have two equal diameters. Finally, for a cubic crystal, all three diameters of the ellipsoid must be equal, it becomes a sphere, and the polarizability of the crystal is the same in all directions.

There is a big game of figuring out the possible kinds of tensors for all the possible symmetries of a crystal. It is called a "group-theoretical" analysis. But for the simple case of the polarizability tensor, it is relatively easy to see what the relations must be.

31–4 Other tensors; the tensor of inertia

There are many other examples of tensors appearing in physics. For example, in a metal, or in any conductor, one often finds that the current density $\boldsymbol{j}$ is approximately proportional to the electric field $\boldsymbol{E}$; the proportionality constant is called the conductivity σ:

$$\boldsymbol{j} = \sigma \boldsymbol{E}.$$

For crystals, however, the relation between $\boldsymbol{j}$ and $\boldsymbol{E}$ is more complicated; the conductivity is not the same in all directions. The conductivity is a tensor, and we write

$$j_i = \sum \sigma_{ij} E_j.$$

Another example of a physical tensor is the moment of inertia. In Chapter 18 of Volume I we saw that a solid object rotating about a fixed axis has an angular momentum L proportional to the angular velocity ω, and we called the proportionality factor I, the moment of inertia:

$$L = I\omega.$$

For an arbitrarily shaped object, the moment of inertia depends on its orientation with respect to the axis of rotation. For instance, a rectangular block will have different moments about each of its three orthogonal axes. Now angular velocity $\boldsymbol{\omega}$ and angular momentum $\boldsymbol{L}$ are both vectors. For rotations about one of the axes of symmetry, they are parallel. But if the moment of inertia is different for the

three principal axes, then ω and L are, in general, not in the same direction (see Fig. 31–4). They are related in a way analogous to the relation between E and P. In general, we must write

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z, \\ L_y &= I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z, \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z. \end{aligned} \tag{31.16}$$

The nine coefficients I_{ij} are called the tensor of inertia. Following the analogy with the polarization, the kinetic energy for any angular momentum must be some quadratic form in the components ω_x, ω_y, and ω_z:

$$\text{KE} = \tfrac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j. \tag{31.17}$$

We can use the energy to define the ellipsoid of inertia. Also, energy arguments can be used to show that the tensor is symmetric—that $I_{ij} = I_{ji}$.

The tensor of inertia for a rigid body can be worked out if the shape of the object is known. We need only to write down the total kinetic energy of all the particles in the body. A particle of mass m and velocity v has the kinetic energy $\frac{1}{2}mv^2$, and the total kinetic energy is just the sum

$$\sum \tfrac{1}{2}mv^2$$

over all of the particles of the body. The velocity v of each particle is related to the angular velocity ω of the solid body. Let's assume that the body is rotating about its center of mass, which we take to be at rest. Then if r is the displacement of a particle from the center of mass, its velocity v is given by $\omega \times r$. So the total kinetic energy is

$$\text{KE} = \sum \tfrac{1}{2}m(\omega \times r)^2. \tag{31.18}$$

Now all we have to do is write $\omega \times r$ out in terms of the components ω_x, ω_y, ω_z, and x, y, z, and compare the result with Eq. (31.17); we find I_{ij} by identifying terms. Carrying out the algebra, we write

$$\begin{aligned} (\omega \times r)^2 &= (\omega \times r)_x^2 + (\omega \times r)_y^2 + (\omega \times r)_z^2 \\ &= (\omega_y z - \omega_z y)^2 + (\omega_z x - \omega_x z)^2 + (\omega_x y - \omega_y x)^2 \\ &= + \omega_y^2 z^2 - 2\omega_y\omega_z zy + \omega_z^2 y^2 \\ &\quad + \omega_z^2 x^2 - 2\omega_z\omega_x xz + \omega_x^2 z^2 \\ &\quad + \omega_x^2 y^2 - 2\omega_x\omega_y yx + \omega_y^2 x^2. \end{aligned}$$

Multiplying this equation by $m/2$, summing over all particles, and comparing with Eq. (31.17), we see that I_{xx}, for instance, is given by

$$I_{xx} = \sum m(y^2 + z^2).$$

This is the formula we have had before (Chapter 19, Vol. I) for the moment of inertia of a body about the x-axis. Since $r^2 = x^2 + y^2 + z^2$, we can also write this term as

$$I_{xx} = \sum m(r^2 - x^2).$$

Working out all of the other terms, the tensor of inertia can be written as

$$I_{ij} = \begin{bmatrix} \sum m(r^2 - x^2) & -\sum mxy & -\sum mxz \\ -\sum myx & \sum m(r^2 - y^2) & -\sum myz \\ -\sum mzx & -\sum mzy & \sum m(r^2 - z^2) \end{bmatrix}. \tag{31.19}$$

If you wish, this may be written in "tensor notation" as

$$I_{ij} = \sum m(r^2\,\delta_{ij} - r_i r_j), \tag{31.20}$$

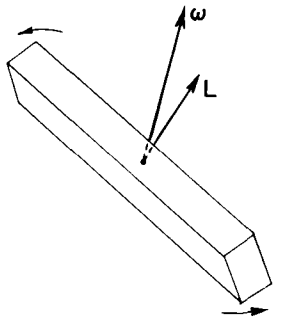

Fig. 31–4. The angular momentum L of a solid object is not, in general, parallel to its angular velocity ω.

where the r_i are the components (x, y, z) of the position vector of a particle and the $\sum$ means to sum over all the particles. The moment of inertia, then, is a tensor of the second rank whose terms are a property of the body and relate $\boldsymbol{L}$ to $\boldsymbol{\omega}$ by

$$L_i = \sum_j I_{ij}\omega_j \tag{31.21}$$

For a body of any shape whatever, we can find the ellipsoid of inertia and, therefore, the three principal axes. Referred to these axes, the tensor will be diagonal, so for any object there are always three orthogonal axes for which the angular velocity and angular momentum are parallel. They are called the principal axes of inertia.

31–5 The cross product

We should point out that we have been using tensors of the second rank since Chapter 20 of Volume I. There, we defined a "torque in a plane," such as τ_{xy}, by

$$\tau_{xy} = xF_y - yF_x.$$

Generalized to three dimensions, we could write

$$\tau_{ij} = r_iF_j - r_jF_i. \tag{31.22}$$

The quantity τ_{ij} is a tensor of the second rank. One way to see that this is so is by combining τ_{ij} with some vector, say the unit vector $\boldsymbol{e}$, according to

$$\sum_j \tau_{ij}e_j.$$

If this quantity is a *vector*, then τ_{ij} must transform as a tensor—this is our definition of a tensor. Substituting for τ_{ij}, we have

$$\sum_j \tau_{ij}e_j = \sum_j r_iF_je_j - \sum_j r_je_jF_i$$

$$= r_i(\boldsymbol{F}\cdot\boldsymbol{e}) - (\boldsymbol{r}\cdot\boldsymbol{e})F_i.$$

Since the dot products are scalars, the two terms on the right-hand side are vectors, and likewise their difference. So τ_{ij} is a tensor.

But τ_{ij} is a special kind of tensor; it is *antisymmetric*, that is,

$$\tau_{ij} = -\tau_{ji},$$

so it has only three nonzero terms—τ_{xy}, τ_{yz}, and τ_{zx}. We were able to show in Chapter 20 of Volume I that these three terms, almost "by accident," transform like the three components of a vector, so that we could *define*

$$\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z) = (\tau_{yz}, \tau_{zx}, \tau_{xy})$$

We say "by accident," because it happens only in three dimensions. In four dimensions, for instance, an antisymmetric tensor of the second rank has *six* nonezero terms and certainly cannot be replaced by a vector with *four* components.

Just as the axial vector $\boldsymbol{\tau} = \boldsymbol{r}\times\boldsymbol{F}$ is a tensor, so also is every cross product of two polar vectors—all the same arguments apply. By luck, however, they are also representable by vectors (really pseudovectors), so our mathematics has been made easier for us.

Mathematically, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are any two vectors, the nine quantities a_ib_j form a tensor (although it may have no useful physical purpose). Thus, for the position vector r_i, r_ir_j is a tensor, and since δ_{ij} is also, we see that the right side of Eq. (31.20) is indeed a tensor. Likewise Eq. (31.22) is a tensor, since the two terms on the right-hand side are tensors.

31–6 The tensor of stress

The symmetric tensors we have described so far arose as coefficients in relating one vector to another. We would like to look now at a tensor which has a different physical significance—the tensor of *stress*. Suppose we have a solid object with various forces on it. We say that there are various "stresses" inside, by which we mean that there are internal forces between neighboring parts of the material. We have talked a little about such stresses in a two-dimensional case when we considered the surface tension in a stretched diaphragm in Section 12–3. We will now see that the internal forces in the material of a three-dimensional body can be described in terms of a tensor.

Consider a body of some elastic material—say a block of jello. If we make a cut through the block, the material on each side of the cut will, in general, get displaced by the internal forces. Before the cut was made, there must have been forces between the two parts of the block that kept the material in place; we can define the stresses in terms of these forces. Suppose we look at an imaginary plane perpendicular to the x-axis—like the plane σ in Fig 31–5—and ask about the force across a small area $\Delta y\,\Delta z$ in this plane The material on the left of the area exerts the force $\Delta\boldsymbol{F}_1$ on the material to the right, as shown in part (b) of the figure There is, of course, the opposite reaction force $-\Delta\boldsymbol{F}_1$ exerted on the material to the left of the surface. If the area is small enough, we expect that $\Delta\boldsymbol{F}_1$ is proportional to the area $\Delta y\,\Delta z$.

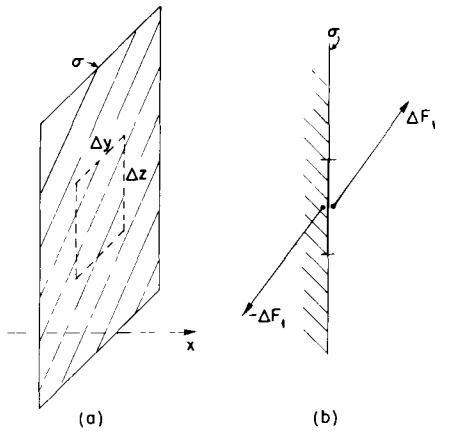

Fig 31–5. The material to the left of the plane σ exerts across the area $\Delta y\,\Delta z$ the force $\Delta\boldsymbol{F}_1$ on the material to the right of the plane.

You are already familiar with one kind of stress—the pressure in a static liquid. There the force is equal to the pressure times the area and is at right angles to the surface element. For solids—also for viscous liquids in motion—the force need not be normal to the surface; there are *shear* forces in addition to pressures (positive or negative) (By a "shear" force we mean the *tangential* components of the force across a surface.) All three components of the force must be taken into account. Notice also that if we make our cut on a plane with some other orientation, the forces will be different. A complete description of the internal stress requires a tensor.

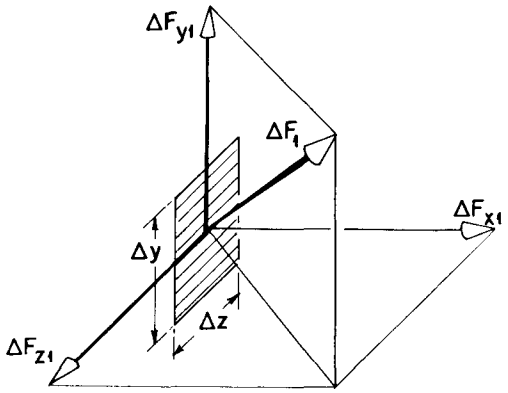

Fig. 31–6. The force $\Delta\boldsymbol{F}_1$ across an element of area $\Delta y\,\Delta z$ perpendicular to the x-axis is resolved into the three components ΔF_{x1}, ΔF_{y1}, and ΔF_{z1}.

We define the stress tensor in the following way: First, we imagine a cut perpendicular to the x-axis and resolve the force $\Delta\boldsymbol{F}_1$ across the cut into its components ΔF_{x1}, ΔF_{y1}, ΔF_{z1}, as in Fig. 31–6. The ratio of these forces to the area $\Delta y\,\Delta z$, we call S_{xx}, S_{yx}, and S_{zx}. For example,

$$S_{yx} = \frac{\Delta F_{y1}}{\Delta y\,\Delta z}.$$

The first index y refers to the direction force component; the second index x is normal to the area. If you wish, you can write the area $\Delta y\,\Delta z$ as Δa_x, meaning an element of area perpendicular to x. Then

$$S_{yx} = \frac{\Delta F_{y1}}{\Delta a_x}.$$

Next, we think of an imaginary cut perpendicular to the y-axis. Across a small

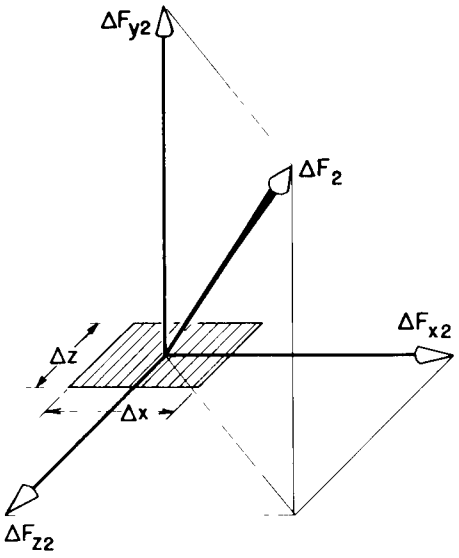

Fig. 31–7. The force across an element of area perpendicular to y is resolved into three rectangular components

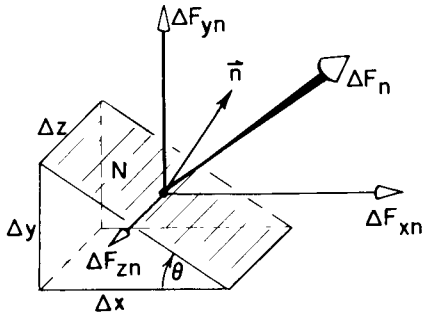

Fig. 31–8. The force $\boldsymbol{F}_n$ across the face N (whose unit normal is $\boldsymbol{n}$) is resolved into components.

area $\Delta x\,\Delta z$ there will be a force $\Delta \boldsymbol{F}_2$. Again we resolve this force into three components, as shown in Fig 31–7, and define the three components of the stress, S_{xy}, S_{yy}, S_{zy}, as the force per unit area in the three directions. Finally, we make an imaginary cut perpendicular to z and define the three components S_{xz}, S_{yz}, and S_{zz}. So we have the nine numbers

$$S_{ij} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix}. \tag{31.23}$$

We want to show now that these nine numbers are sufficient to describe completely the internal state of stress, and that S_{ij} is indeed a tensor Suppose we want to know the force across a surface oriented at some arbitrary angle Can we find it from S_{ij}? Yes, in the following way: We imagine a little solid figure which has one face N in the new surface, and the other faces parallel to the coordinate axes. If the face N happened to be parallel to the z-axis, we would have the triangular piece shown in Fig. 31–8. (This is a somewhat special case, but will illustrate well enough the general method.) Now the stress forces on the little solid triangle in Fig 31–8 are in equilibrium (at least in the limit of infinitesimal dimensions), so the total force on it must be zero. We know the forces on the faces parallel to the coordinate axes directly from S_{ij} Their vector sum must equal the force on the face N, so we can express this force in terms of S_{ij}.

Our assumption that the *surface* forces on the small triangular volume are in equilibrium neglects any other *body* forces that might be present, such as gravity or pseudo forces if our coordinate system is not an inertial frame Notice, however, that such body forces will be proportional to the *volume* of the little triangle and, therefore, to Δx, Δy, Δz, whereas all the surface forces are proportional to the areas such as $\Delta x\,\Delta y$, $\Delta y\,\Delta z$, etc. So if we take the scale of the little wedge small enough, the body forces can always be neglected in comparison with the surface forces.

Let's now add up the forces on the little wedge. We take first the x-component, which is the sum of five parts—one from each face However, if Δz is small enough, the forces on the triangular faces (perpendicular to the z-axis) will be equal and opposite, so we can forget them. The x-component of the force on the bottom rectangle is

$$\Delta F_{x2} = S_{xy}\,\Delta x\,\Delta z.$$

The x-component of the force on the vertical rectangle is

$$\Delta F_{x1} = S_{xx}\,\Delta y\,\Delta z.$$

These two must be equal to the x-component of the force *outward* across the face N. Let's call $\boldsymbol{n}$ the unit vector normal to the face N, and the force on it $\boldsymbol{F}_n$, then we have

$$\Delta F_{xn} = S_{xx}\,\Delta y\,\Delta z + S_{xy}\,\Delta x\,\Delta z.$$

The x-component S_{xn} of the stress across this plane is equal to ΔF_{xn} divided by the area, which is $\Delta z\sqrt{\Delta x^2 + \Delta y^2}$, or

$$S_{xn} = S_{xx}\frac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} + S_{xy}\frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}}.$$

Now $\Delta x/\sqrt{\Delta x^2 + \Delta y^2}$ is the cosine of the angle θ between $\boldsymbol{n}$ and the y-axis, as shown in Fig. 31–8, so it can also be written as n_y, the y-component of $\boldsymbol{n}$. Similarly, $\Delta y/\sqrt{\Delta x^2 + \Delta y^2}$ is $\sin\theta = n_x$. We can write

$$S_{xn} = S_{xx}n_x + S_{xy}n_y.$$

If we now generalize to an arbitrary surface element, we would get that

$$S_{xn} = S_{xx}n_x + S_{xy}n_y + S_{xz}n_z$$

or, in general,

$$S_{in} = \sum_j S_{ij} n_j. \tag{31.24}$$

We *can* find the force across any surface element in terms of the S_{ij}, so it does describe completely the state of internal stress of the material.

Equation (31 24) says that the tensor S_{ij} relates the force $\boldsymbol{S}_n$ to the unit vector $\boldsymbol{n}$, just as α_{ij} relates $\boldsymbol{P}$ to $\boldsymbol{E}$. Since $\boldsymbol{n}$ and $\boldsymbol{S}_n$ are vectors, the components of S_{ij} must transform as a tensor with changes in coordinate axes. So S_{ij} is indeed a tensor.

We can also show that S_{ij} is a *symmetric* tensor by looking at the forces on a little cube of material. Suppose we take a little cube, oriented with its faces parallel to our coordinate axes, and look at it in cross section, as shown in Fig 31–9. If we let the edge of the cube be one unit, the x- and y-components of the forces on the faces normal to the x- and y-axes might be as shown in the figure. If the cube is small, the stresses do not change appreciably from one side of the cube to the opposite side, so the force components are equal and opposite as shown Now there must be no torque on the cube, or it would start spinning. The total torque about the center is $(S_{yx} - S_{xy})$ (times the unit edge of the cube), and since the total is zero, S_{yx} is equal to S_{xy}, and the stress tensor is symmetric.

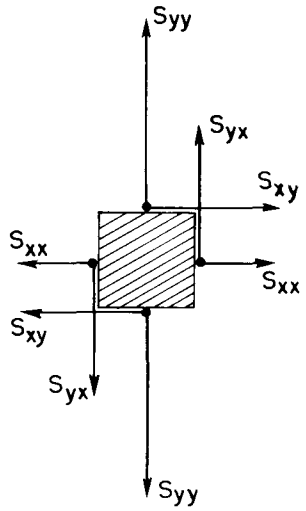

Fig. 31–9. The x- and y-forces on four faces of a small unit cube.

Since S_{ij} is a symmetric tensor, it can be described by an ellipsoid which will have three principal axes. For surfaces normal to these axes, the stresses are particularly simple—they correspond to pushes or pulls perpendicular to the surfaces There are no shear forces along these faces. For *any* stress, we can always choose our axes so that the shear components are zero. If the ellipsoid is a sphere, there are only normal forces in *any* direction. This corresponds to a hydrostatic pressure (positive or negative). So for a hydrostatic pressure, the tensor is diagonal and all three components are equal; they are, in fact, just equal to the pressure p. We can write

$$S_{ij} = p\delta_{ij}. \tag{31.25}$$

The stress tensor—and also its ellipsoid—will, in general, vary from point to point in a block of material; to describe the whole block we need to give the value of each component of S_{ij} as a function of position. So the stress tensor is a *field*. We have had *scalar fields*, like the temperature $T(x, y, z)$, which give one number for each point in space, and *vector fields* like $\boldsymbol{E}(x, y, z)$, which give three numbers for each point. Now we have a *tensor field* which gives nine numbers for each point in space—or really six for the symmetric tensor S_{ij}. A complete description of the internal forces in an arbitrarily distorted solid requires six functions of x, y, and z.

31–7 Tensors of higher rank

The stress tensor S_{ij} describes the internal *forces* of matter. If the material is elastic, it is convenient to describe the internal *distortion* in terms of another tensor T_{ij}—called the *strain* tensor. For a simple object like a bar of metal, you know that the change in length, ΔL, is approximately proportional to the force, so we say it obeys Hooke's law:

$$\Delta L = \gamma F.$$

For a solid elastic body with arbitrary distortions, the strain T_{ij} is related to the stress S_{ij} by a set of linear equations:

$$T_{ij} = \sum_{k,l} \gamma_{ijkl} S_{kl}. \tag{31.26}$$

Also, you know that the potential energy of a spring (or bar) is

$$\tfrac{1}{2} F\,\Delta L = \tfrac{1}{2}\gamma F^2.$$

The generalization for the elastic energy *density* in a solid body is

$$U_{\text{elastic}} = \sum_{ijkl} \tfrac{1}{2}\gamma_{ijkl} S_{ij} S_{kl}. \tag{31.27}$$

The complete description of the elastic properties of a crystal must be given in terms of the coefficients γ_{ijkl}. This introduces us to a new beast. It is a tensor of the *fourth* rank. Since each index can take on any one of three values, x, y, or z, there are $3^4 = 81$ coefficients. But there are really only 21 *different* numbers. First, since S_{ij} is symmetric, it has only six different values, and only 36 *different* coefficients are needed in Eq. (31.27). But also, S_{ij} can be interchanged with S_{kl} without changing the energy, so γ_{ijkl} must be symmetric if we interchange ij and kl. This reduces the number of different coefficients to 21. So to describe the elastic properties of a crystal of the lowest possible symmetry requires 21 elastic constants! This number is, of course, reduced for crystals of higher symmetry. For example, a cubic crystal has only three elastic constants, and an isotropic substance has only two.

That the latter is true can be seen as follows How can the components of γ_{ijkl} be independent of the direction of the axes, as they must be if the material is isotropic? *Answer:* They can be independent *only* if they are expressible in terms of the tensor δ_{ij}. There are two possible expressions, $\delta_{ij}\delta_{kl}$ and $\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$, which have the required symmetry, so γ_{ijkl} must be a linear combination of them. Therefore, for isotropic materials,

$$\gamma_{ijkl} = a(\delta_{ij}\delta_{kl}) + b(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

and the material requires two constants, a and b, to describe its elastic properties. We will leave it for you to show that a cubic crystal needs only three

As a final example, this time of a third-rank tensor, we have the piezoelectric effect. Under stress, a crystal generates an electric field proportional to the stress; hence, in general, the law is

$$E_i = \sum_{j,k} P_{ijk}S_{jk},$$

where E_i is the electric field, and the P_{ijk} are the piezoelectric coefficients—or the piezoelectric tensor Can you show that if the crystal has a center of inversion (invariant under $x, y, z \to -x, -y, -z$) the piezoelectric coefficients are all zero?

31–8 The four-tensor of electromagnetic momentum

All the tensors we have looked at so far in this chapter relate to the three dimensions of space; they are defined to have a certain transformation property under spatial rotations. In Chapter 26 we had occasion to use a tensor in the four dimensions of relativistic space-time—the electromagnetic field tensor $F_{\mu\nu}$ The components of such a four-tensor transform under a Lorentz transformation of the coordinates in a special way that we worked out. (Although we did not do it that way, we could have considered the Lorentz transformation as a "rotation" in a four-dimensional "space" called Minkowski space; then the analogy with what we are doing here would have been clearer)

As our last example, we want to consider another tensor in the four dimensions (t, x, y, z) of relativity theory. When we wrote the stress tensor, we defined S_{ij} as a component of a force across a unit area. But a force is equal to the time rate of change of a momentum. Therefore, instead of saying "S_{xy} is the x-component of the force across a unit area perpendicular to y," we could equally well say, "S_{xy} is the rate of flow of the x-component of momentum through a unit area perpendicular to y." In other words, each term of S_{ij} also represents the flow of the i-component of momentum through a unit area perpendicular to the j-direction These are pure space components, but they are parts of a "larger" tensor $S_{\mu\nu}$ in four dimensions (μ and $\nu = t, x, y, z$) containing additional components like S_{tx}, S_{yt}, S_{tt}, etc. We will now try to find the physical meaning of these extra components.

We know that the space components represent flow of momentum. We can get a clue on how to extend this to the time dimension by studying another kind of "flow"—the flow of electric charge. For the *scalar* quantity, charge, the rate of flow (per unit area perpendicular to the flow) is a space *vector*—the current density

vector $\boldsymbol{j}$. We have seen that the time component of this flow vector is the density of the stuff that is flowing For instance, $\boldsymbol{j}$ can be combined with a time component, $j_t = \rho$, the charge density, to make the four-vector $j_\mu = (\rho, \boldsymbol{j})$; that is, the μ in j_μ takes on the values t, x, y, z to mean "density, rate of flow in the x-direction, rate of flow in y, rate of flow in z" of the scalar charge.

Now by analogy with our statement about the time component of the flow of a scalar quantity, we might expect that with S_{xx}, S_{xy}, and S_{xz}, describing the flow of the x-component of momentum, there should be a time component S_{xt} which would be the density of whatever is flowing; that is, S_{xt} should be the density of x-momentum. So we can extend our tensor horizontally to include a t-component We have

$$S_{xt} = \text{density of } x\text{-momentum},$$

$$S_{xx} = x\text{-flow of } x\text{-momentum},$$

$$S_{xy} = y\text{-flow of } x\text{-momentum},$$

$$S_{xz} = z\text{-flow of } x\text{-momentum}.$$

Similarly, for the y-component of momentum we have the three components of flow—S_{yx}, S_{yy}, S_{yz}—to which we should add a fourth term:

$$S_{yt} = \text{density of } y\text{-momentum}.$$

And, of course, to S_{zx}, S_{zy}, S_{zz} we would add

$$S_{zt} = \text{density of } z\text{-momentum}.$$

In four dimensions there is also a t-component of momentum, which is, we know, energy So the tensor S_{ij} should be extended vertically with S_{tx}, S_{ty}, and S_{tz}, where

$$\begin{aligned} S_{tx} &= x\text{-flow of energy}, \\ S_{ty} &= y\text{-flow of energy}, \\ S_{tz} &= z\text{-flow of energy}; \end{aligned} \tag{31.28}$$

that is, S_{tx} is the flow of energy per unit area and per unit time across a surface perpendicular to the x-axis, and so on. Finally, to complete our tensor we need S_{tt}, which would be the *density* of *energy*. We have extended our stress tensor S_{ij} of three dimensions to the four-dimensional *stress-energy tensor* $S_{\mu\nu}$. The index μ can take on the four values t, x, y, and z, meaning, respectively, "density," "flow per unit area in the x-direction," "flow per unit area in the y-direction," and "flow per unit area in the z-direction " In the same way, ν takes on the four values t, x, y, z to tell us *what* flows, namely, "energy," "momentum in the x-direction," "momentum in the y-direction," and "momentum in the z-direction."

As an example, we will discuss this tensor not in matter, but in a region of free space in which there is an electromagnetic field. We know that the flow of energy is the Poynting vector $\boldsymbol{S} = \epsilon_0 c^2 \boldsymbol{E} \times \boldsymbol{B}$. So the x-, y-, and z-components of $\boldsymbol{S}$ are, from the relativistic point of view, the components S_{tx}, S_{ty}, and S_{tz} of our four-dimensional stress-energy tensor. The symmetry of the tensor S_{ij} carries over into the time components as well, so the four-dimensional tensor $S_{\mu\nu}$ is symmetric:

$$S_{\mu\nu} = S_{\nu\mu}. \tag{31.29}$$

In other words, the components S_{xt}, S_{yt}, S_{zt}, which are the *densities* of x, y, and z *momentum*, are also equal to the x-, y-, and z-components of the Poynting vector $\boldsymbol{S}$, the *energy flow*—as we have already shown in an earlier chapter by a different kind of argument.

The remaining components of the electromagnetic stress tensor $S_{\mu\nu}$ can also be expressed in terms of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ That is to say, we must admit stress or, to put it less mysteriously, flow of momentum in the electromagnetic field We discussed this in Chapter 27 in connection with Eq (27.21), but did not work out the details

Those who want to exercise their prowess in tensors in four dimensions might like to see the formula for $S_{\mu\nu}$ in terms of the fields:

$$S_{\mu\nu} = \frac{\epsilon_0}{2}\left(\sum_{\alpha} F_{\mu\alpha}F_{\nu\alpha} - \tfrac{1}{4}\,\delta_{\mu\nu}\sum_{\alpha,\beta} F_{\beta\alpha}F_{\beta\alpha}\right),$$

where sums on α, β are on t, x, y, z but (as usual in relativity) we adopt a special meaning for the sum sign $\sum$ and for the symbol δ. In the sums the x, y, z terms are to be *subtracted* and $\delta_{tt} = +1$, while $\delta_{xx} = \delta_{yy} = \delta_{zz} = -1$ and $\delta_{\mu\nu} = 0$ for $\mu \neq \nu$ ($c = 1$). Can you verify that it gives the energy density $S_{tt} = (\epsilon_0/2)\,(E^2 + B^2)$ and the Poynting vector $\epsilon_0 \boldsymbol{E} \times \boldsymbol{B}$? Can you show that in an electrostatic field with $\boldsymbol{B} = 0$ the principal axes of stress are in the direction of the electric field, that there is a *tension* $(\epsilon_0/2)E^2$ along the direction of the field, and that there is an equal *pressure* in directions perpendicular to the field direction?