

Electrodynamics in Relativistic Notation

25-1 Four-vectors

We now discuss the application of the special theory of relativity to electrodynamics. Since we have already studied the special theory of relativity in Chapters 15 through 17 of Vol. I, we will just review quickly the basic ideas.

It is found experimentally that the laws of physics are unchanged if we move with uniform velocity. You can't tell if you are inside a spaceship moving with uniform velocity in a straight line, unless you look outside the spaceship, or at least make an observation having to do with the world outside. Any true law of physics we write down must be arranged so that this fact of nature is built in.

The relationship between the space and time of two systems of coordinates, one, S' , in uniform motion in the x -direction with speed v relative to the other, S , is given by the *Lorentz transformation*:

$$\begin{aligned} t' &= \frac{t - vx}{\sqrt{1 - v^2}}, & y' &= y, \\ x' &= \frac{x - vt}{\sqrt{1 - v^2}}, & z' &= z. \end{aligned} \quad (25.1)$$

The laws of physics must be such that after a Lorentz transformation, the new form of the laws looks just like the old form. This is just like the principle that the laws of physics don't depend on the *orientation* of our coordinate system. In Chapter 11 of Vol. I, we saw that the way to describe mathematically the invariance of physics with respect to rotations was to write our equations in terms of *vectors*.

For example, if we have two vectors

$$\mathbf{A} = (A_x, A_y, A_z) \quad \text{and} \quad \mathbf{B} = (B_x, B_y, B_z),$$

we found that the combination

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

was not changed if we transformed to a rotated coordinate system. So we know that if we have a scalar product like $\mathbf{A} \cdot \mathbf{B}$ on both sides of an equation, the equation will have exactly the same form in all rotated coordinate systems. We also discovered an operator (see Chapter 2),

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

which, when applied to a scalar function, gave three quantities which transform just like a vector. With this operator we defined the gradient, and in combination with other vectors, the divergence and the Laplacian. Finally we discovered that by taking sums of certain products of pairs of the components of two vectors we could get three new quantities which behaved like a new vector. We called it the *cross product* of two vectors. Using the cross product with our operator ∇ we then defined the curl of a vector.

Since we will be referring back to what we have done in vector analysis, we have put in Table 25-1 a summary of all the important vector operations in three dimensions that we have used in the past. The point is that it must be possible to write the equations of physics so that both sides transform the same way under

25-1 Four-vectors

25-2 The scalar product

25-3 The four-dimensional gradient

25-4 Electrodynamics in four-dimensional notation

25-5 The four-potential of a moving charge

25-6 The invariance of the equations of electrodynamics

In this chapter: $c = 1$

Review: Chapter 15, Vol. I, *The Special Theory of Relativity*
 Chapter 16, Vol. I, *Relativistic Energy and Momentum*
 Chapter 17, Vol. I, *Space-Time*
 Chapter 13, Vol. II, *Magnetostatics*

Table 25-1

The important quantities and operations of vector analysis in three dimensions

Definition of a vector	$A = (A_x, A_y, A_z)$
Scalar product	$A \cdot B$
Differential vector operator	∇
Gradient	$\nabla\phi$
Divergence	$\nabla \cdot A$
Laplacian	$\nabla \cdot \nabla = \nabla^2$
Cross product	$A \times B$
Curl	$\nabla \times A$

rotations. If one side is a vector, the other side must also be a vector, and both sides will change together in exactly the same way if we rotate our coordinate system. Similarly, if one side is a scalar, the other side must also be a scalar, so that neither side changes when we rotate coordinates, and so on.

Now in the case of special relativity, time and space are inextricably mixed, and we must do the analogous things for four dimensions. We want our equations to remain the same not only for rotations, but also for *any* inertial frame. That means that our equations should be invariant under the Lorentz transformation of equations (25.1). The purpose of this chapter is to show you how that can be done. Before we get started, however, we want to do something that makes our work a lot easier (and saves some confusion). And that is to choose our units of length and time so that the speed of light c is equal to 1. You can think of it as taking our unit of time to be *the time that it takes light to go one meter* (which is about 3×10^{-9} sec). We can even call this time unit "one meter." Using this unit, all of our equations will show more clearly the space-time symmetry. Also, all the c 's will disappear from our relativistic equations. (If this bothers you, you can always put the c 's back into any equation by replacing every t by ct , or, in general, by sticking in a c wherever it is needed to make the dimensions of the equations come out right.) With this groundwork we are ready to begin. Our program is to do in the four dimensions of space-time all of the things we did with vectors for three dimensions. It is really quite a simple game, we just work by analogy. The only real complications is the notation (we've already used up the vector symbol for three dimensions) and one slight twist of signs.

First, by analogy with vectors in three dimensions, we define a *four-vector* as a set of the four quantities $a_t, a_x, a_y,$ and a_z , which transform like $t, x, y,$ and z when we change to a moving coordinate system. There are several different notations people use for a four-vector; we will write a_μ , by which we mean the group of four numbers (a_t, a_x, a_y, a_z) —in other words, the subscript μ can take on the four "values" t, x, y, z . It will also be convenient, at times, to indicate the three space components by a three-vector, like this: $a_\mu = (a_t, \mathbf{a})$.

We have already encountered one four-vector, which consists of the energy and momentum of a particle (Chapter 17, Vol. I). In our new notation we write

$$p_\mu = (E, \mathbf{p}), \tag{25.2}$$

which means that the four-vector p_μ is made up of the energy E and the three components of the three-vector \mathbf{p} of a particle.

It looks as though the game is really very simple—for each three-vector in physics all we have to do is find what the remaining component should be, and we have a four-vector. To see that this is not the case, consider the velocity vector with components

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}.$$

The question is: What is the time component? Instinct should give the right answer. Since four-vectors are like t, x, y, z , we would guess that the time component is

$$v_t = \frac{dt}{dt} = 1.$$

This is wrong. The reason is that the t in each denominator is not an invariant when we make a Lorentz transformation. The numerators have the right behavior to make a four-vector, but the dt in the denominator spoils things; it is unsymmetric and is not the same in two different systems.

It turns out that the four "velocity" components which we have written down will become the components of a four-vector if we just divide by $\sqrt{1 - v^2}$. We can see that that is true because if we start with the momentum four-vector

$$p_\mu = (E, \mathbf{p}) = \left(\frac{m_0}{\sqrt{1 - v^2}}, \frac{m_0 \mathbf{v}}{\sqrt{1 - v^2}} \right), \tag{25.3}$$

and divide it by the rest mass m_0 , which is an invariant scalar in *four dimensions*, we have

$$\frac{p_\mu}{m_0} = \left(\frac{1}{\sqrt{1-v^2}}, \frac{\mathbf{v}}{\sqrt{1-v^2}} \right), \quad (24.4)$$

which must still be a four-vector. (Dividing by an *invariant scalar* doesn't change the transformation properties) So we can *define* the “velocity four-vector” u_μ by

$$\begin{aligned} u_t &= \frac{1}{\sqrt{1-v^2}}, & u_y &= \frac{v_y}{\sqrt{1-v^2}}, \\ u_x &= \frac{v_x}{\sqrt{1-v^2}}, & u_z &= \frac{v_z}{\sqrt{1-v^2}}, \end{aligned} \quad (25.5)$$

The four-velocity is a useful quantity; we can, for instance, write

$$p_\mu = m_0 u_\mu. \quad (25.6)$$

This is the typical sort of form an equation which is relativistically correct must have; each side is a four-vector. (The right-hand side is an invariant times a four-vector, which is still a four-vector.)

25-2 The scalar product

It is an accident of life, if you wish, that under coordinate rotations the distance of a point from the origin does not change. This means mathematically that $r^2 = x^2 + y^2 + z^2$ is an invariant. In other words, after a rotation $r'^2 = r^2$, or

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2.$$

Now the question is: Is there a similar quantity which is invariant under the Lorentz transformation? There is. From Eq. (25.1) you can see that

$$t'^2 - x'^2 = t^2 - x^2$$

That is pretty nice, except that it depends on a particular choice of the x -direction. We can fix that up by subtracting y^2 and z^2 . Then any Lorentz transformation *plus* a rotation will leave the quantity unchanged. So the quantity which is analogous to r^2 for four dimensions, in three dimensions is

$$t^2 - x^2 - y^2 - z^2.$$

It is an invariant under what is called the “complete Lorentz group”—which means for transformation of both translations at constant velocity *and* rotations.

Now since this invariance is an algebraic matter depending only on the transformation rules of Eq. (25.1)—plus rotations—it is true for any four-vector (by definition they all transform the same). So for a four-vector a_μ we have that

$$a_t'^2 - a_x'^2 - a_y'^2 - a_z'^2 = a_t^2 - a_x^2 - a_y^2 - a_z^2.$$

We will call this quantity the square of “the length” of the four-vector a_μ . (Sometimes people change the sign of all the terms and call the length $a_x^2 + a_y^2 + a_z^2 - a_t^2$, so you'll have to watch out.)

Now if we have *two* vectors a_μ and b_μ , their corresponding components transform in the same way, so the combination

$$a_t b_t - a_x b_x - a_y b_y - a_z b_z$$

is also an invariant (scalar) quantity. (We have in fact already proved this in Chapter 17 of Vol. I.) Clearly this expression is quite analogous to the dot product for vectors. We will, in fact, call it the *dot product* or *scalar product* of two four-vectors. It would seem logical to write it as $a_\mu \cdot b_\mu$, so it would *look* like a dot product. But, unhappily, it's not done that way; it is usually written without the dot.

So we will follow the convention and write the dot product simply as $a_\mu b_\mu$. So, by definition,

$$a_\mu b_\mu = a_t b_t - a_x b_x - a_y b_y - a_z b_z \quad (25.7)$$

Whenever you see two identical subscripts together (we will occasionally have to use ν or some other letter instead of μ) it means that you are to take the four products and sum, remembering the minus sign for the products of the space components. With this convention the invariance of the scalar product under a Lorentz transformation can be written as

$$a'_\mu b'_\mu = a_\mu b_\mu.$$

Since the last three terms in (25.7) are just the scalar dot product in three dimensions, it is often more convenient to write

$$a_\mu b_\mu = a_t b_t - \mathbf{a} \cdot \mathbf{b}.$$

It is also obvious that the four-dimensional length we described above can be written as $a_\mu a_\mu$:

$$a_\mu a_\mu = a_t^2 - a_x^2 - a_y^2 - a_z^2 = a_t^2 - \mathbf{a} \cdot \mathbf{a}. \quad (25.8)$$

It will also be convenient to sometimes write this quantity as a_μ^2 :

$$a_\mu^2 \equiv a_\mu a_\mu.$$

We will now give you an illustration of the usefulness of four-vector dot products. Antiprotons (\bar{P}) are produced in large accelerators by the reaction



That is, an energetic proton collides with a proton at rest (for example, in a hydrogen target placed in the beam), and if the incident proton has enough energy, a proton-antiproton pair may be produced, in addition to the two original protons.* The question is: How much energy must be given to the incident proton to make this reaction energetically possible?

The easiest way to get the answer is to consider what the reaction looks like in the center-of-mass (CM) system (see Fig. 25-1). We'll call the incident proton a and its four-momentum p_μ^a . Similarly, we'll call the target proton b and its four-

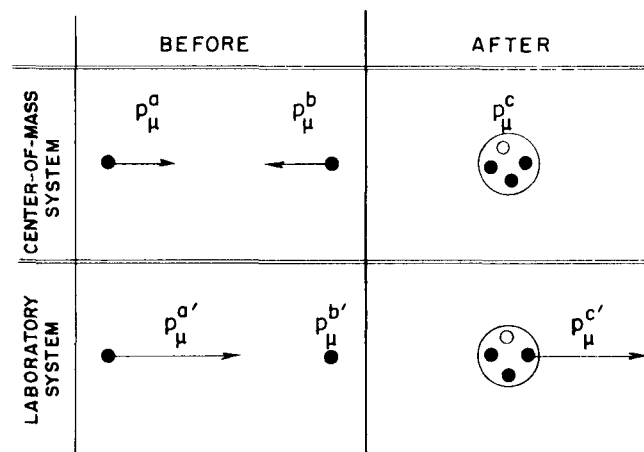
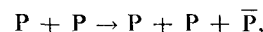
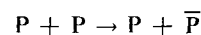


Fig. 25-1. The reaction $P + P \rightarrow 3P + \bar{P}$ viewed in the laboratory and CM systems. The incident proton is supposed to have just barely enough energy to make the reaction go. Protons are denoted by solid circles; antiprotons, by open circles.

* You may well ask: Why not consider the reactions



or even



which clearly require less energy? The answer is that a principle called *conservation of baryons* tells us the quantity "number of protons minus number of antiprotons" cannot change. This quantity is 2 on the left side of our reaction. Therefore, if we want an antiproton on the right side, we must have also *three* protons (or other baryons).

momentum p_μ^b . If the incident proton has *just barely* enough energy to make the reaction go, the final state—the situation after the collision—will consist of a glob containing three protons and an antiproton at rest in the CM system. If the incident energy were slightly higher, the final state particles would have some kinetic energy and be moving apart; if the incident energy were slightly lower, there would not be enough energy to make the four particles.

If we call p_μ^c the total four-momentum of the whole glob in the final state, conservation of energy and momentum tells us that

$$\mathbf{p}^a + \mathbf{p}^b = \mathbf{p}^c,$$

and

$$E^a + E^b = E^c.$$

Combining these two equations, we can write that

$$p_\mu^a + p_\mu^b = p_\mu^c. \quad (25.9)$$

Now the important thing is that this is an equation among four-vectors, and is, therefore, true in any inertial frame. We can use this fact to simplify our calculations. We start by taking the “length” of each side of Eq. (25.9); they are, of course, also equal. We get

$$(p_\mu^a + p_\mu^b)(p_\mu^a + p_\mu^b) = p_\mu^c p_\mu^c. \quad (25.10)$$

Since $p_\mu^c p_\mu^c$ is invariant, we can evaluate it in any coordinate system. In the CM system, the time component of p_μ^c is the rest energy of four protons, namely $4M$, and the space part \mathbf{p} is zero; so $p_\mu^c = (4M, \mathbf{0})$. We have used the fact that the rest mass of an antiproton equals the rest mass of a proton, and we have called this common mass M .

Thus, Eq. (25.10) becomes

$$p_\mu^a p_\mu^a + 2p_\mu^a p_\mu^b + p_\mu^b p_\mu^b = 16M^2. \quad (25.11)$$

Now $p_\mu^a p_\mu^a$ and $p_\mu^b p_\mu^b$ are very easy, since the “length” of the momentum four-vector of any particle is just the mass of the particle squared:

$$p_\mu p_\mu = E^2 - \mathbf{p}^2 = M^2.$$

This can be shown by direct calculation or, more cleverly, by noting that for a particle *at rest* $p_\mu = (M, \mathbf{0})$, so $p_\mu p_\mu = M^2$. But since it is an invariant, it is equal to M^2 in *any* frame. Using these results in Eq. (25.11), we have

$$2p_\mu^a p_\mu^b = 14M^2$$

or

$$p_\mu^a p_\mu^b = 7M^2. \quad (25.12)$$

Now we can also evaluate $p_\mu^a p_\mu^b$ in the laboratory system. The four-vector p_μ^a can be written (E^a, \mathbf{p}^a) , while $p_\mu^b = (M, \mathbf{0})$, since it describes a proton at rest. Thus, $p_\mu^a p_\mu^b$ must also be equal to ME^a , and since we know the scalar product is an invariant this must be numerically the same as what we found in (25.12). So we have that

$$E^a = 7M,$$

which is the result we were after. The *total* energy of the initial proton must be at least $7M$ (about 6.6 Gev since $M = 938$ Mev) or, subtracting the rest mass M , the *kinetic* energy must be at least $6M$ (about 5.6 Gev). The Bevatron accelerator at Berkeley was designed to give about 6.2 Gev of kinetic energy to the protons it accelerates, in order to be able to make antiprotons.

Since scalar products are invariant, they are always interesting to evaluate. What about the “length” of the four-velocity $u_\mu u_\mu$?

$$u_\mu u_\mu = u_t^2 - \mathbf{u}^2 = \frac{1}{1 - v^2} - \frac{v^2}{1 - v^2} = 1.$$

Thus, u_μ is the *unit four-vector*

25-3 The four-dimensional gradient

The next thing that we have to discuss is the four-dimensional analog of the gradient. We recall (Chapter 14, Vol. I) that the three differential operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ transform like a three-vector and are called the gradient. The same scheme ought to work in four dimensions; that is, we might guess that the four-dimensional gradient should be $(\partial/\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)$. *This is wrong.*

To see the error, consider a scalar function ϕ which depends only on x and t . The change in ϕ , if we make a small change Δt in t while holding x constant, is

$$\Delta\phi = \frac{\partial\phi}{\partial t} \Delta t. \quad (25.13)$$

On the other hand, according to a moving observer,

$$\Delta\phi = \frac{\partial\phi}{\partial x'} \Delta x' + \frac{\partial\phi}{\partial t'} \Delta t'.$$

We can express $\Delta x'$ and $\Delta t'$ in terms of Δt by using Eq (25.1). Remembering that we are holding x constant, so that $\Delta x = 0$, we write

$$\Delta x' = -\frac{v}{\sqrt{1-v^2}} \Delta t; \quad \Delta t' = \frac{\Delta t}{\sqrt{1-v^2}}.$$

Thus,

$$\begin{aligned} \Delta\phi &= \frac{\partial\phi}{\partial x'} \left(-\frac{v}{\sqrt{1-v^2}} \Delta t \right) + \frac{\partial\phi}{\partial t'} \left(\frac{\Delta t}{\sqrt{1-v^2}} \right) \\ &= \left(\frac{\partial\phi}{\partial t'} - v \frac{\partial\phi}{\partial x'} \right) \frac{\Delta t}{\sqrt{1-v^2}}. \end{aligned}$$

Comparing this result with Eq. (25.13), we learn that

$$\frac{\partial\phi}{\partial t} = \frac{1}{\sqrt{1-v^2}} \left(\frac{\partial\phi}{\partial t'} - v \frac{\partial\phi}{\partial x'} \right) \quad (25.14)$$

A similar calculation gives

$$\frac{\partial\phi}{\partial x} = \frac{1}{\sqrt{1-v^2}} \left(\frac{\partial\phi}{\partial x'} - v \frac{\partial\phi}{\partial t'} \right). \quad (25.15)$$

Now we can see that the gradient is rather strange. The formulas for x and t in terms of x' and t' [obtained by solving Eq. (25.1)] are:

$$t = \frac{t' + vx'}{\sqrt{1-v^2}}, \quad x = \frac{x' + vt'}{\sqrt{1-v^2}}.$$

This is the way a four-vector *must* transform. But Eqs. (25.14) and (25.15) have a couple of signs wrong!

The answer is that instead of the *incorrect* $(\partial/\partial t, \nabla)$, we must *define* the *four-dimensional gradient* operator, which we will call ∇_μ , by

$$\nabla_\mu = \left(\frac{\partial}{\partial t}, -\nabla \right) = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right). \quad (25.16)$$

With this definition, the sign difficulties encountered above go away, and ∇_μ behaves as a four-vector should. (It's rather awkward to have those minus signs, but that's the way the world is.) Of course, what it means to say that ∇_μ "behaves like a four-vector" is simply that the four-gradient of a scalar is a four-vector. If ϕ is a true scalar invariant field (Lorentz invariant) then $\nabla_\mu\phi$ is a four-vector field.

All right, now that we have vectors, gradients, and dot products, the next thing is to look for an invariant which is analogous to the divergence of three-dimensional vector analysis. Clearly, the analog is to form the expression $\nabla_\mu b_\mu$, where b_μ is a four-vector field whose components are functions of space and time.

We define the divergence of the four-vector $b_\mu = (b_t, \mathbf{b})$ as the dot product of ∇_μ and b_μ :

$$\begin{aligned}\nabla_\mu b_\mu &= \frac{\partial}{\partial t} b_t - \left(-\frac{\partial}{\partial x}\right) b_x - \left(-\frac{\partial}{\partial y}\right) b_y - \left(-\frac{\partial}{\partial z}\right) b_z \\ &= \frac{\partial}{\partial t} b_t + \nabla \cdot \mathbf{b},\end{aligned}\tag{25.17}$$

where $\nabla \cdot \mathbf{b}$ is the ordinary three-divergence of the three-vector \mathbf{b} . Note that one has to be careful with the signs. Some of the minus signs come from the definition of the scalar product, Eq. (25.7); the others are required because the space components of ∇_μ are $-\partial/\partial x$, etc., as in Eq. (25.16). The divergence as defined by (25.17) is an invariant and gives the same answer in all coordinate systems which differ by a Lorentz transformation.

Let's look at a physical example in which the four-divergence shows up. We can use it to solve the problem of the fields around a moving wire. We have already seen (Section 13-7) that the electric charge density ρ and the current density \mathbf{j} form a four-vector $j_\mu = (\rho, \mathbf{j})$. If an uncharged wire carries the current j_x , then in a frame moving past it with velocity v (along x), the wire will have the charge and current density [obtained from the Lorentz transformation Eqs. (25.1)] as follows:

$$\rho' = \frac{-vj_x}{\sqrt{1-v^2}}, \quad j'_x = \frac{j_x}{\sqrt{1-v^2}}.$$

These are just what we found in Chapter 13. We can then use these sources in Maxwell's equation in *the moving system* to find the fields.

The charge conservation law, Section 13-2, also takes on a simple form in the four-vector notation. Consider the four-divergence of j_μ :

$$\nabla_\mu j_\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}.\tag{25.18}$$

The law of the conservation of charge says that the outflow of current per unit volume must equal the negative rate of increase of charge density. In other words, that

$$\nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t}.$$

Putting this into Eq. (25.18), the law of conservation of charge takes on the simple form

$$\nabla_\mu j_\mu = 0.\tag{25.19}$$

Since $\nabla_\mu j_\mu$ is an invariant scalar, if it is zero in one frame it is zero in all frames. We have the result that if charge is conserved in one coordinate system, it is conserved in all coordinate systems moving with uniform velocity.

As our last example we want to consider the scalar product of the gradient operator ∇_μ with itself. In three dimensions, such a product gives the Laplacian

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

What do we get in four dimensions? That's easy. Following our rules for dot products and gradients, we get

$$\begin{aligned}\nabla_\mu \nabla_\mu &= \frac{\partial}{\partial t} \frac{\partial}{\partial t} - \left(-\frac{\partial}{\partial x}\right) \left(-\frac{\partial}{\partial x}\right) - \left(-\frac{\partial}{\partial y}\right) \left(-\frac{\partial}{\partial y}\right) - \left(-\frac{\partial}{\partial z}\right) \left(-\frac{\partial}{\partial z}\right) \\ &= \frac{\partial^2}{\partial t^2} - \nabla^2.\end{aligned}$$

This operator, which is the analog of the three-dimensional Laplacian, is called

the *D'Alembertian* and has a special notation:

$$\square^2 = \nabla_\mu \nabla_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2. \quad (25.20)$$

From its definition it is an invariant scalar operator; if it operates on a four-vector field, it produces a new four-vector field. (Some people define the *D'Alembertian* with the opposite sign to Eq. (25.20), so you will have to be careful when reading the literature.)

We have now found four-dimensional equivalents of most of the three-dimensional quantities we had listed in Table 25-1. (We do not yet have the equivalents of the cross product and the curl operation; we won't get to them until the next chapter.) It may help you remember how they go if we put all the important definitions and results together in one place, so we have made such a summary in Table 25-2.

Table 25-2

The important quantities of vector analysis in three and four dimensions.

	Three dimensions	Four dimensions
Vector	$\mathbf{A} = (A_x, A_y, A_z)$	$a_\mu = (a_t, a_x, a_y, a_z) = (a_t, \mathbf{a})$
Scalar product	$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$	$a_\mu b_\mu = a_t b_t - a_x b_x - a_y b_y - a_z b_z = a_t b_t - \mathbf{a} \cdot \mathbf{b}$
Vector operator	$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$	$\nabla_\mu = (\partial/\partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z) = (\partial/\partial t, -\nabla)$
Gradient	$\nabla\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z} \right)$	$\nabla_\mu\varphi = \left(\frac{\partial\varphi}{\partial t}, -\frac{\partial\varphi}{\partial x}, -\frac{\partial\varphi}{\partial y}, -\frac{\partial\varphi}{\partial z} \right) = \left(\frac{\partial\varphi}{\partial t}, \nabla\varphi \right)$
Divergence	$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla_\mu a_\mu = \frac{\partial a_t}{\partial t} + \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = \frac{\partial a_t}{\partial t} + \nabla \cdot \mathbf{a}$
Laplacian and <i>D'Alembertian</i>	$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$	$\nabla_\mu \nabla_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial t^2} - \nabla^2 = \square^2$

25-4 Electrodynamics in four-dimensional notation

We have already encountered the *D'Alembertian* operator, without giving it that name, in Section 18-6; the differential equations we found there for the potentials can be written in the new notations as:

$$\square^2 \phi = \frac{\rho}{\epsilon_0}, \quad \square^2 \mathbf{A} = \frac{\mathbf{j}}{\epsilon_0}. \quad (25.21)$$

The four quantities on the right-hand side of the two equations in (25.21) are ρ , j_x , j_y , j_z , divided by ϵ_0 , which is a universal constant which will be the same in all coordinate systems if the same unit of charge is used in all frames. So the four quantities ρ/ϵ_0 , j_x/ϵ_0 , j_y/ϵ_0 , j_z/ϵ_0 also transform as a four-vector. We can write them as J_μ/ϵ_0 . The *D'Alembertian* doesn't change when the coordinate system is changed, so the quantities ϕ , A_x , A_y , A_z must also transform like a four-vector—which means that they *are* the components of a four-vector. In short,

$$A_\mu = (\phi, \mathbf{A})$$

is a four-vector. What we call the scalar and vector potentials are really different aspects of the same physical thing. They belong together. And if they are kept together the relativistic invariance of the world is obvious. We call A_μ the *four-potential*.

In the four-vector notation Eqs. (25.21) become simply

$$\square^2 A_\mu = \frac{j_\mu}{\epsilon_0}, \quad (25.22)$$

The physics of this equation is just the same as Maxwell's equations. But there is some pleasure in being able to rewrite them in an elegant form. The pretty form is also meaningful, it shows directly the invariance of electrodynamics under the Lorentz transformation.

Remember that Eqs (25.21) could be deduced from Maxwell's equations only if we imposed the gauge condition

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0, \quad (25.23)$$

which just says $\nabla_\mu A_\mu = 0$; the gauge condition says that the divergence of the four-vector A_μ is zero. This condition is called the *Lorentz condition*. It is very convenient because it is an invariant condition and therefore Maxwell's equations stay in the form of Eq. (25.22) for all frames.

25-5 The four-potential of a moving charge

Although it is implicit in what we have already said, let us write down the transformation laws which give ϕ and \mathbf{A} in a moving system in terms of ϕ and \mathbf{A} in a stationary system. Since $A_\mu = (\phi, \mathbf{A})$ is a four-vector, the equations must look just like Eqs. (25.1), except that t is replaced by ϕ , and \mathbf{x} is replaced by \mathbf{A} . Thus,

$$\begin{aligned} \phi' &= \frac{\phi - vA_x}{\sqrt{1 - v^2}}, & A'_y &= A_y, \\ A'_x &= \frac{A_x - v\phi}{\sqrt{1 - v^2}}, & A'_z &= A_z. \end{aligned} \quad (25.24)$$

This assumes that the primed coordinate system is moving with speed v in the positive x -direction, as measured in the unprimed coordinate system.

We will consider one example of the usefulness of the idea of the four-potential. What are the vector and scalar potentials of a charge q moving with speed v along the x -axis? The problem is easy in a coordinate system moving with the charge, since in this system the charge is standing still. Let's say that the charge is at the origin of the S' -frame, as shown in Fig. 25-2. The scalar potential in the moving system is then given by

$$\phi' = \frac{q}{4\pi\epsilon_0 r'}, \quad (25.25)$$

r' being the distance from q to the field point, as measured in the moving system. The vector potential \mathbf{A}' is, of course, zero.

Now it is straightforward to find ϕ and \mathbf{A} , the potentials as measured in the stationary coordinates. The inverse relations to Eqs. (25.24) are

$$\begin{aligned} \phi &= \frac{\phi' + vA'_x}{\sqrt{1 - v^2}}, & A_y &= A'_y, \\ A_x &= \frac{A'_x + v\phi'}{\sqrt{1 - v^2}}, & A_z &= A'_z. \end{aligned} \quad (25.26)$$

Using the ϕ' given by Eq. (25.25), and $\mathbf{A}' = 0$, we get

$$\begin{aligned} \phi &= \frac{q}{4\pi\epsilon_0} \frac{1}{r' \sqrt{1 - v^2}} \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{1 - v^2} \sqrt{x'^2 + y'^2 + z'^2}}. \end{aligned}$$

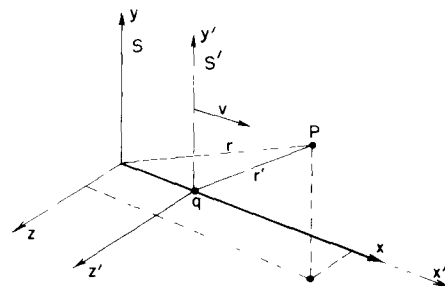


Fig. 25-2. The frame S' moves with velocity v (in the x -direction) with respect to S . A charge at rest at the origin of S' is at $x = vt$ in S . The potentials at P can be computed in either frame.

This gives us the scalar potential ϕ we would see in S , but, unfortunately, expressed in terms of the S' coordinates. We can get things in terms of t, x, y, z by substituting for t', x', y' , and z' , using (25.1). We get

$$\phi = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{1-v^2}} \frac{1}{\sqrt{[(x-vt)/\sqrt{1-v^2}]^2 + y^2 + z^2}}. \quad (25.27)$$

Following the same procedure for the components of A , you can show that

$$A = v\phi. \quad (25.28)$$

These are the same formulas we derived by a different method in Chapter 21.

25-6 The invariance of the equations of electrodynamics

We have found that the potentials ϕ and A taken together form a four-vector which we call A_μ , and that the wave equations—the full equations which determine the A_μ in terms of the j_μ —can be written as in Eq. (25.22). This equation, together with the conservation of charge, Eq. (25.19), gives us the fundamental law of the electromagnetic field:

$$\square^2 A_\mu = \frac{1}{\epsilon_0} j_\mu, \quad \nabla_\mu j_\mu = 0. \quad (25.29)$$

There, in one tiny space on the page, are all of the Maxwell equations—beautiful and simple. Did we learn anything from writing the equations this way, besides that they are beautiful and simple? In the first place, is it anything different from what we had before when we wrote everything out in all the various components? Can we from this equation deduce something that could not be deduced from the wave equations for the potentials in terms of the charges and currents? The answer is definitely no. The only thing we have been doing is changing the names of things—using a new notation. We have written a square symbol to represent the derivatives, but it still means nothing more nor less than the second derivative with respect to t , minus the second derivative with respect to x , minus the second derivative with respect to y , minus the second derivative with respect to z . And the μ means that we have four equations, one each for $\mu = t, x, y, \text{ or } z$. What then is the significance of the fact that the equations can be written in this simple form? From the point of view of deducing anything directly, it doesn't mean anything. Perhaps, though, the simplicity of the equations means that nature also has a certain simplicity.

Let us show you something interesting that we have recently discovered: *All of the laws of physics can be contained in one equation.* That equation is

$$U = 0. \quad (25.30)$$

What a simple equation! Of course, it is necessary to know what the symbol means. U is a physical quantity which we will call the “unworldliness” of the situation. And we have a formula for it. Here is how you calculate the unworldliness. You take all of the known physical laws and write them in a special form. For example, suppose you take the law of mechanics, $F = ma$, and rewrite it as $F - ma = 0$. Then you can call $(F - ma)$ —which should, of course, be zero—the “mismatch,” of mechanics. Next, you take the *square* of this mismatch and call it U_1 , which can be called the “unworldliness of mechanical effects.” In other words, you take

$$U_1 = (F - ma)^2. \quad (25.31)$$

Now you write another physical law, say, $\nabla \cdot E = \rho/\epsilon_0$ and define

$$U_2 = \left(\nabla \cdot E - \frac{\rho}{\epsilon_0} \right)^2,$$

which you might call “the gaussian unworldliness of electricity.” You continue to write U_3, U_4 , and so on—one for every physical law there is

Finally you call the *total* unworldliness U of the world the sum of the various unworldlinesses U_i from all the subphenomena that are involved; that is, $U = \sum U_i$. Then the great “law of nature” is

$$\boxed{U = 0.} \quad (25.32)$$

This “law” means, of course, that the sum of the squares of all the individual mismatches is zero, and the only way the sum of a lot of squares can be zero is for each one of the terms to be zero

So the “beautifully simple” law in Eq. (25.32) is equivalent to the whole series of equations that you originally wrote down. It is therefore absolutely obvious that a simple notation that just hides the complexity in the definitions of symbols is not real simplicity. *It is just a trick.* The beauty that appears in Eq. (25.32)—just from the fact that several equations are hidden within it—is no more than a trick. When you unwrap the whole thing, you get back where you were before.

However, there *is* more to the simplicity of the laws of electromagnetism written in the form of Eq. (25.29). It means more, just as a theory of vector analysis means more. The fact that the electromagnetic equations can be written in a very particular notation *which was designed* for the four-dimensional geometry of the Lorentz transformations—in other words, as a vector equation in the four-space—means that it is invariant under the Lorentz transformations. It is because the Maxwell equations are invariant under those transformations that they can be written in a beautiful form.

It is no accident that the equations of electrodynamics can be written in the beautifully elegant form of Eq. (25.29). The theory of relativity was developed *because it was found experimentally* that the phenomena predicted by Maxwell’s equations were the same in all inertial systems. And it was precisely by studying the transformation properties of Maxwell’s equations that Lorentz discovered his transformation as the one which left the equations invariant.

There is, however, another reason for writing our equations this way. It has been discovered—after Einstein guessed that it might be so—that *all* of the laws of physics are invariant under the Lorentz transformation. That is the principle of relativity. Therefore, if we invent a notation which shows immediately when a law is written down whether it is invariant or not, we can be sure that in trying to make new theories we will write only equations which are consistent with the principle of relativity.

The fact that the Maxwell equations are simple in this particular notation is not a miracle, because the notation was invented with them in mind. But the interesting physical thing is that *every law* of physics—the propagation of meson waves or the behavior of neutrinos in beta decay, and so forth—must have this same invariance under the same transformation. Then when you are moving at a uniform velocity in a spaceship, all of the laws of nature transform together in such a way that no new phenomenon will show up. It is because the principle of relativity is a fact of nature that in the notation of four-dimensional vectors the equations of the world will look simple.