The Maxwell Equations

18-1 Maxwell's equations

In this chapter we come back to the complete set of the four Maxwell equations that we took as our starting point in Chapter 1. Until now, we have been studying Maxwell's equations in bits and pieces; it is time to add one final piece, and to put them all together. We will then have the complete and correct story for electromagnetic fields that may be changing with time in any way. Anything said in this chapter that contradicts something said earlier is true and what was said earlier is false—because what was said earlier applied to such special situations as, for instance, steady currents or fixed charges. Although we have been very careful to point out the restrictions whenever we wrote an equation, it is easy to forget all of the qualifications and to learn too well the wrong equations. Now we are ready to give the whole truth, with no qualifications (or almost none).

The complete Maxwell equations are written in Table 18-1, in words as well as in mathematical symbols. The fact that the words are equivalent to the equations should by this time be familiar—you should be able to translate back and forth from one form to the other.

The first equation—that the divergence of E is the charge density over ϵ_0—is true in general. In dynamic as well as in static fields, Gauss' law is always valid. The flux of E through any closed surface is proportional to the charge inside. The third equation is the corresponding general law for magnetic fields. Since there are no magnetic charges, the flux of B through any closed surface is always zero. The second equation, that the curl of E is $-\partial B/\partial t$, is Faraday's law and was discussed in the last two chapters. It also is generally true. The last equation has something new. We have seen before only the part of it which holds for steady currents. In that case we said that the curl of B is $j/\epsilon_0 c^2$, but the correct general equation has a new part that was discovered by Maxwell.

Until Maxwell's work, the known laws of electricity and magnetism were those we have studied in Chapters 3 through 17. In particular, the equation for the magnetic field of steady currents was known only as

$$\nabla \times B = \frac{j}{\epsilon_0 c^2}.$$

(18.1)

Maxwell began by considering these known laws and expressing them as differential equations, as we have done here. (Although the ∇ notation was not yet invented, it is mainly due to Maxwell that the importance of the combinations of derivatives, which we today call the curl and the divergence, first became apparent.) He then noticed that there was something strange about Eq. (18.1). If one takes the divergence of this equation, the left-hand side will be zero, because the divergence of a curl is always zero. So this equation requires that the divergence of j also be zero. But if the divergence of j is zero, then the total flux of current out of any closed surface is also zero.

The flux of current from a closed surface is the decrease of the charge inside the surface. This certainly cannot in general be zero because we know that the charges can be moved from one place to another. The equation

$$\nabla \cdot j = -\frac{\partial \rho}{\partial t}$$

(18.2)

has, in fact, been almost our definition of j. This equation expresses the very funda-

Table 18-1 Classical Physics

Maxwell's equations

I. $\nabla \cdot E = \dfrac{\rho}{\epsilon_0}$ (Flux of E through a closed surface) = (Charge inside)/ϵ_0

II. $\nabla \times E = -\dfrac{\partial B}{\partial t}$ (Line integral of E around a loop) = $-\dfrac{d}{dt}$ (Flux of B through the loop)

III. $\nabla \cdot B = 0$ (Flux of B through a closed surface) = 0

IV. $c^2 \nabla \times B = \dfrac{j}{\epsilon_0} + \dfrac{\partial E}{\partial t}$ c^2(Integral of B around a loop) = (Current through the loop)/ϵ_0

$+ \dfrac{\partial}{\partial t}$ (Flux of E through the loop)

$$\left[\begin{array}{l} \text{Conservation of charge} \\[2mm] \nabla \cdot j = -\dfrac{\partial \rho}{\partial t} \qquad \text{(Flux of current through a closed surface)} = -\dfrac{\partial}{\partial t}\text{(Charge inside)} \end{array}\right]$$

Force law

$F = q(E + v \times B)$

Law of motion

$\dfrac{d}{dt}(p) = F,$ where $p = \dfrac{mv}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation

$F = -G \dfrac{m_1 m_2}{r^2} e_r$

---

mental law that electric charge is conserved—any flow of charge must come from some supply. Maxwell appreciated this difficulty and proposed that it could be avoided by adding the term $\partial E/\partial t$ to the right-hand side of Eq. (18.1); he then got the fourth equation in Table 18-1:

IV. $c^2 \nabla \times B = \dfrac{j}{\epsilon_0} + \dfrac{\partial E}{\partial t}.$

It was not yet customary in Maxwell's time to think in terms of abstract fields. Maxwell discussed his ideas in terms of a model in which the vacuum was like an elastic solid. He also tried to explain the meaning of his new equation in terms of the mechanical model. There was much reluctance to accept his theory, first because of the model, and second because there was at first no experimental justification. Today, we understand better that what counts are the equations themselves and not the model used to get them. We may only question whether the equations are true or false. This is answered by doing experiments, and untold numbers of experiments have confirmed Maxwell's equations. If we take away the scaffolding he used to build it, we find that Maxwell's beautiful edifice stands on its own. He brought together all of the laws of electricity and magnetism and made one complete and beautiful theory.

Let us show that the extra term is just what is required to straighten out the difficulty Maxwell discovered. Taking the divergence of his equation (IV in Table 18-1), we must have that the divergence of the right-hand side is zero:

$$\nabla \cdot \frac{j}{\epsilon_0} + \nabla \cdot \frac{\partial E}{\partial t} = 0. \tag{18.3}$$

In the second term, the order of the derivatives with respect to coordinates and time can be reversed, so the equation can be rewritten as

$$\nabla \cdot \boldsymbol{j} + \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{E} = 0. \tag{18.4}$$

But the first of Maxwell's equations says that the divergence of E is ρ/ϵ_0. Inserting this equality in Eq. (18.4), we get back Eq. (18.2), which we know is true. Conversely, if we accept Maxwell's equations—and we do because no one has ever found an experiment that disagrees with them—we must conclude that charge is always conserved.

The laws of physics have no answer to the question: "What happens if a charge is suddenly created at this point—what electromagnetic effects are produced?" No answer can be given because our equations say it doesn't happen. If it *were* to happen, we would need new laws, but we cannot say what they would be. We have not had the chance to observe how a world without charge conservation behaves. According to our equations, if you suddenly place a charge at some point, you had to carry it there from somewhere else. In that case, we can say what would happen.

When we added a new term to the equation for the curl of E, we found that a whole new class of phenomena was described. We shall see that Maxwell's little addition to the equation for $\nabla \times \boldsymbol{B}$ also has far-reaching consequences. We can touch on only a few of them in this chapter.

18–2 How the new term works

As our first example we consider what happens with a spherically symmetric radial distribution of current. Suppose we imagine a little sphere with radioactive material on it. This radioactive material is squirting out some charged particles. (Or we could imagine a large block of jello with a small hole in the center into which some charge had been injected with a hypodermic needle and from which the charge is slowly leaking out.) In either case we would have a current that is everywhere radially outward. We will assume that it has the same magnitude in all directions.

Let the total charge inside any radius r be $Q(r)$. If the radial current density at the same radius is $j(r)$, then Eq. (18.2) requires that Q decreases at the rate

$$\frac{\partial Q(r)}{\partial t} = -4\pi r^2 j(r). \tag{18.5}$$

We now ask about the magnetic field produced by the currents in this situation. Suppose we draw some loop Γ on a sphere of radius r, as shown in Fig. 18–1. There is some current through this loop, so we might expect to find a magnetic field circulating in the direction shown.

But we are already in difficulty. How can the B have any particular direction on the sphere? A different choice of Γ would allow us to conclude that its direction is exactly opposite to that shown. So how *can* there be any circulation of B around the currents?

We are saved by Maxwell's equation. The circulation of B depends not only on the total *current* through Γ but also on the rate of change with time of the *electric flux* through it. It must be that these two parts just cancel. Let's see if that works out.

The electric field at the radius r must be $Q(r)/4\pi\epsilon_0 r^2$—so long as the charge is symmetrically distributed, as we assume. It is radial, and its rate of change is then

$$\frac{\partial E}{\partial t} = \frac{1}{4\pi\epsilon_0 r^2} \frac{\partial Q}{\partial t}. \tag{18.6}$$

Comparing this with Eq. (18.5), we see that at any radius

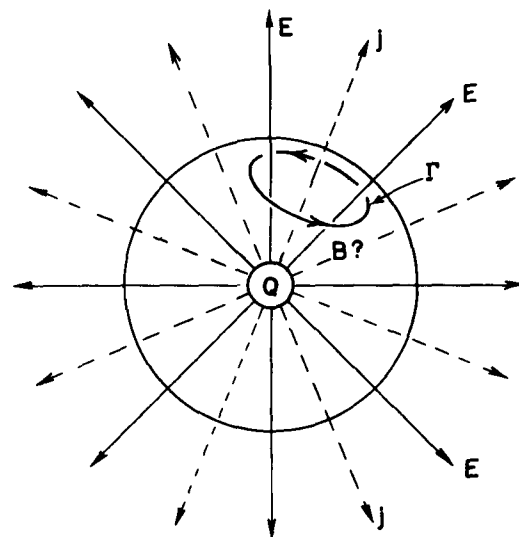$$\frac{\partial E}{\partial t} = -\frac{j}{\epsilon_0}. \tag{18.7}$$

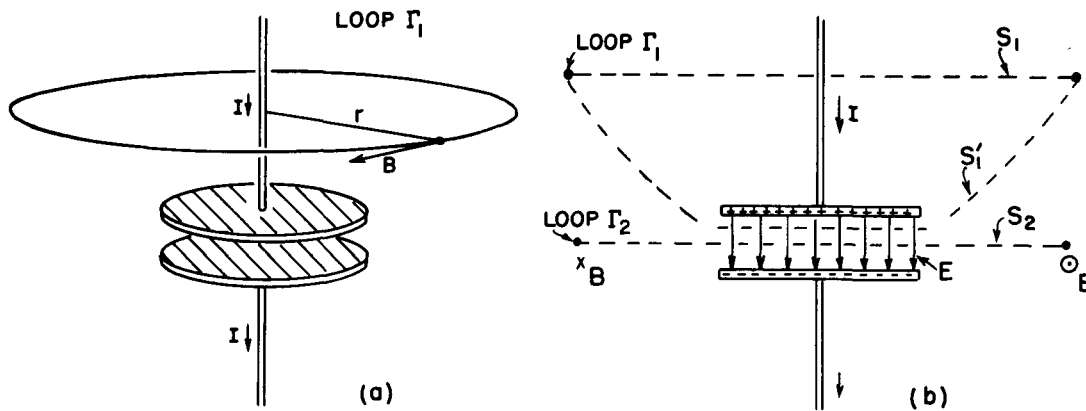Fig. 18–1. What is the magnetic field of a spherically symmetric current?

18–3

Fig. 18–2. The magnetic field near a charging capacitor.

In Eq. IV the two source terms cancel and the curl of B is always zero. There is no magnetic field in our example.

As our second example, we consider the magnetic field of a wire used to charge a parallel-plate condenser (see Fig. 18–2). If the charge Q on the plates is changing with time (but not too fast), the current in the wires is equal to dQ/dt. We would expect that this current will produce a magnetic field that encircles the wire. Surely, the current close to the wire must produce the normal magnetic field—it cannot depend on where the current is going.

Suppose we take a loop Γ_1 which is a circle with radius r, as shown in part (a) of the figure. The line integral of the magnetic field should be equal to the current I divided by $\epsilon_0 c^2$. We have

$$2\pi rB = \frac{I}{\epsilon_0 c^2}. \tag{18.8}$$

This is what we would get for a steady current, but it is also correct with Maxwell's addition, because if we consider the plane surface S inside the circle, there are no electric fields on it (assuming the wire to be a very good conductor). The surface integral of $\partial E/\partial t$ is zero.

Suppose, however, that we now slowly move the curve Γ downward. We get always the same result until we draw even with the plates of the condenser. Then the current I goes to zero. Does the magnetic field disappear? That would be quite strange. Let's see what Maxwell's equation says for the curve Γ_2, which is a circle of radius r whose plane passes between the condenser plates [Fig. 18–2(b)]. The line integral of B around Γ_2 is $2\pi rB$. This must equal the time derivative of the flux of E through the plane circular surface S_2. This flux of E, we know from Gauss' law, must be equal to $1/\epsilon_0$ times the charge Q on one of the condenser plates. We have

$$c^2\, 2\pi rB = \frac{d}{dt}\left(\frac{Q}{\epsilon_0}\right). \tag{18.9}$$

That is very convenient. It is the same result we found in Eq. (18.8). Integrating over the changing electric field gives the same magnetic field as does integrating over the current in the wire. Of course, that is just what Maxwell's equation says. It is easy to see that this must always be so by applying our same arguments to the two surfaces S_1 and S_1' that are bounded by the same circle Γ_1 in Fig. 18–2(b). Through S_1 there is the current I, but no electric flux. Through S_1' there is no current, but an electric flux changing at the rate I/ϵ_0. The same B is obtained if we use Eq. IV with either surface.

From our discussion so far of Maxwell's new term, you may have the impression that it doesn't add much—that it just fixes up the equations to agree with what we already expect. It is true that if we just consider Eq. IV by itself, nothing particularly new comes out. The words "by itself" are, however, all-important. Maxwell's small change in Eq. IV, when combined with the other equations, does indeed produce much that is new and important. Before we take up these matters, however, we want to speak more about Table 18–1.

18–4

18-3 All of classical physics

In Table 18–1 we have all that was known of fundamental *classical* physics, that is, the physics that was known by 1905. Here it all is, in one table. With these equations we can understand the complete realm of classical physics.

First we have the Maxwell equations—written in both the expanded form and the short mathematical form. Then there is the conservation of charge, which is even written in parentheses, because the moment we have the complete Maxwell equations, we can deduce from them the conservation of charge. So the table is even a little redundant. Next, we have written the force law, because having all the electric and magnetic fields doesn't tell us anything until we know what they do to charges. Knowing E and B, however, we can find the force on an object with the charge q moving with velocity v. Finally, having the force doesn't tell us anything until we know what happens when a force pushes on something; we need the law of motion, which is that the force is equal to the rate of change of the momentum. (Remember? We had that in Volume I.) We even include relativity effects by writing the momentum as $p = m_0 v/\sqrt{1 - v^2/c^2}$.

If we really want to be complete, we should add one more law—Newton's law of gravitation—so we put that at the end.

Therefore in one small table we have all the fundamental laws of classical physics—even with room to write them out in words and with some redundancy. This is a great moment. We have climbed a great peak. We are on the top of K-2—we are nearly ready for Mount Everest, which is quantum mechanics. We have climbed the peak of a "Great Divide," and now we can go down the other side.

We have mainly been trying to learn how to understand the equations. Now that we have the whole thing put together, we are going to study what the equations mean—what new things they say that we haven't already seen. We've been working hard to get up to this point. It has been a great effort, but now we are going to have nice coasting downhill as we see all the consequences of our accomplishment.

18-4 A travelling field

Now for the new consequences. They come from putting together all of Maxwell's equations. First, let's see what would happen in a circumstance which we pick to be particularly simple. By assuming that all the quantities vary only in one coordinate, we will have a one-dimensional problem. The situation is shown in Fig. 18–3. We have a sheet of charge located on the *yz*-plane. The sheet is first at rest, then instantaneously given a velocity u in the *y*-direction, and kept moving with this constant velocity. You might worry about having such an "infinite" acceleration, but it doesn't really matter; just imagine that the velocity is brought to u very quickly. So we have suddenly a surface current J (J is the current per unit
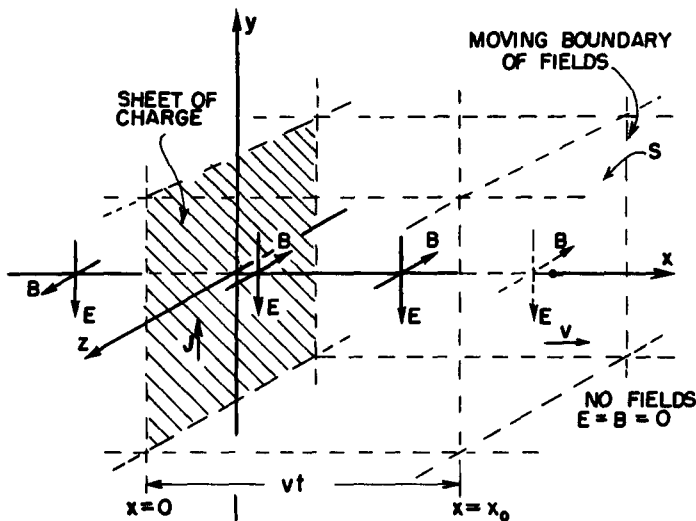

Fig. 18–3. An infinite sheet of charge is suddenly set into motion parallel to itself. There are magnetic and electric fields that propagate out from the sheet at a constant speed.

18-5

Fig. 18-4. (a) The magnitude of B (or E) as a function of x at the time t after the charge sheet is set in motion. (b) The fields for a charge sheet set in motion, toward negative y at $t = T$. (c) The sum of (a) and (b).

width in the z-direction). To keep the problem simple, we suppose that there is also a stationary sheet of charge of opposite sign superposed on the yz-plane, so that there are no electrostatic effects. Also, although in the figure we show only what is happening in a finite region, we imagine that the sheet extends to infinity in $\pm y$ and $\pm z$. In other words, we have a situation where there is no current, and then suddenly there is a uniform sheet of current. What will happen?

Well, when there is a sheet of current in the plus y-direction, there is, as we know, a magnetic field generated which will be in the minus z-direction for $x > 0$ and in the opposite direction for $x < 0$. We could find the magnitude of B by using the fact that the line integral of the magnetic field will be equal to the current over $\epsilon_0 c^2$. We would get that $B = J/2\epsilon_0 c^2$ (since the current I in a strip of width w is Jw and the line integral of B is $2Bw$).

This gives us the field next to the sheet—for small x—but since we are imagining an infinite sheet, we would expect the same argument to give the magnetic field farther out for larger values of x. However, that would mean that the moment we turn on the current, the magnetic field is suddenly changed from zero to a finite value everywhere. But wait! If the magnetic field is suddenly changed, it will produce tremendous electrical effects. (If it changes in *any* way, there are electrical effects.) So because we moved the sheet of charge, we make a changing magnetic field, and therefore electric fields must be generated. If there are electric fields generated, they had to start from zero and change to something else. There will be some $\partial E/\partial t$ that will make a contribution, together with the current J, to the production of the magnetic field. So through the various equations there is a big intermixing, and we have to try to solve for all the fields at once.

By looking at the Maxwell equations alone, it is not easy to see directly how to get the solution. So we will first show you what the answer is and then verify that it does indeed satisfy the equations. The answer is the following: The field B that we computed is, in fact, generated right next to the current sheet (for small x). It must be so, because if we make a tiny loop around the sheet, there is no room for any electric flux to go through it. But the field B out farther—for larger x—is, at first, zero. It stays zero for awhile, and then suddenly turns on. In short, we turn on the current and the magnetic field immediately next to it turns on to a constant value B; then the turning on of B spreads out from the source region. After a certain time, there is a uniform magnetic field everywhere out to some value x, and then zero beyond. Because of the symmetry, it spreads in both the plus and minus x-directions.

The E-field does the same thing. Before $t = 0$ (when we turn on the current), the field is zero everywhere. Then after the time t, both E and B are uniform out to the distance $x = vt$, and zero beyond. The fields make their way forward like a tidal wave, with a front moving at a uniform velocity which turns out to be c, but for a while we will just call it v. A graph of the magnitude of E or B versus x, as they appear at the time t, is shown in Fig. 18-4(a). Looking again at Fig. 18-3, at the time t, the region between $x = \pm vt$ is "filled" with the fields, but they have not yet reached beyond. We emphasize again that we are assuming that the current sheet and, therefore the fields E and B, extend infinitely far in both the y- and z-directions. (We cannot draw an infinite sheet, so we have shown only what happens in a finite area.)

We want now to analyze quantitatively what is happening. To do that, we want to look at two cross-sectional views, a top view looking down along the y-axis, as shown in Fig. 18-5, and a side view looking back along the z-axis, as shown in Fig. 18-6. Suppose we start with the side view. We see the charged sheet moving up; the magnetic field points into the page for $+x$, and out of the page for $-x$, and the electric field is downward everywhere—out to $x = \pm vt$.

Let's see if these fields are consistent with Maxwell's equations. Let's first draw one of those loops that we use to calculate a line integral, say the rectangle Γ_2 shown in Fig. 18-6. You notice that one side of the rectangle is in the region where there are fields, but one side is in the region the fields have still not reached. There is some magnetic flux through this loop. If it is changing, there should be an emf around it. If the wavefront is moving, we will have a changing magnetic
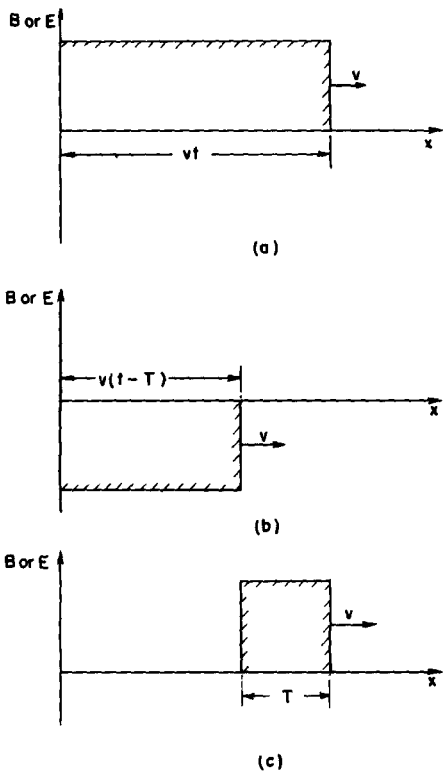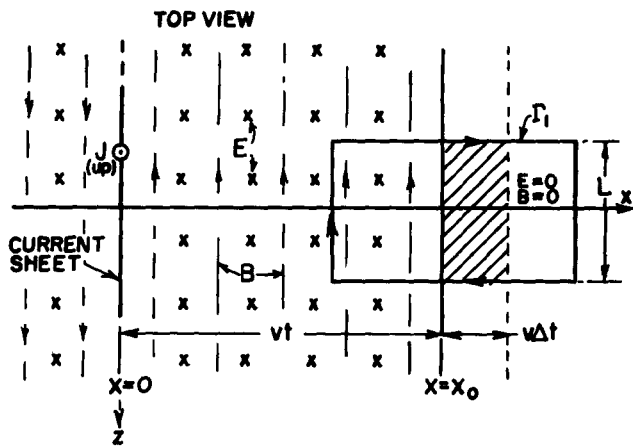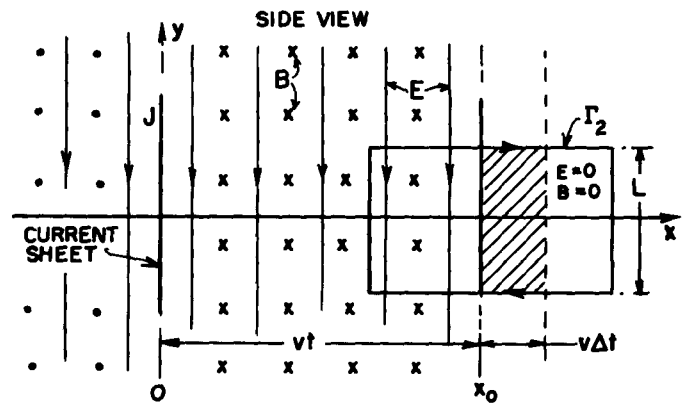
18-6

Fig. 18-5. Top view of Fig. 18-3.

Fig. 18-6. Side view of Fig. 18-3.

flux, because the area in which B exists is progressively increasing at the velocity v. The flux inside Γ_2 is B times the part of the area inside Γ_2 which has a magnetic field. The rate of change of the flux, since the magnitude of B is constant, is the magnitude times the rate of change of the area. The rate of change of the area is easy. If the width of the rectangle Γ_2 is L, the area in which B exists changes by $Lv \, \Delta t$ in the time Δt. (See Fig. 18-6.) The rate of change of flux is then BLv. According to Faraday's law, this should equal the line integral of E around Γ_2, which is just EL. We have the equation

$$E = vB. \tag{18.10}$$

So if the ratio of E to B is v, the fields we have assumed will satisfy Faraday's equation.

But that is not the only equation; we have the other equation relating E and B:

$$c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}. \tag{18.11}$$

To apply this equation, we look at the top view in Fig. 18-5. We have seen that this equation will give us the value of B next to the current sheet. Also, for any loop drawn outside the sheet but behind the wavefront, there is no curl of B nor any j or changing E, so the equation is correct there. Now let's look at what happens for the curve Γ_1 that intersects the wavefront, as shown in Fig. 18-5. Here there are no currents, so Eq. (18.11) can be written—in integral form—as

$$c^2 \oint_{\Gamma_1} B \cdot ds = \frac{d}{dt} \int_{\text{inside } \Gamma_1} E \cdot n \, da. \tag{18.12}$$

The line integral of B is just B times L. The rate of change of the flux of E is due only to the advancing wavefront. The area inside Γ_1, where E is not zero, is increasing at the rate vL. The right-hand side of Eq. (18.12) is then vLE. That equation becomes

$$c^2 B = Ev. \tag{18.13}$$

We have a solution in which we have a constant B and a constant E behind the front, both at right angles to the direction in which the front is moving and at right angles to each other. Maxwell's equations specify the ratio of E to B. From Eqs. (18.10) and (18.13),

$$E = vB, \quad \text{and} \quad E = \frac{c^2}{v} B.$$

But one moment! We have found *two different* conditions on the ratio E/B. Can such a field as we describe really exist? There is, of course, only one velocity v for which both of these equations can hold, namely $v = c$. The wavefront must travel with the velocity c. We have an example in which the electrical influence from a current propagates at a certain finite velocity c.

Now let's ask what happens if we suddenly stop the motion of the charged sheet after it has been on for a short time T. We can see what will happen by the principle of superposition. We had a current that was zero and then was suddenly turned on. We know the solution for that case. Now we are going to add another set of fields. We take another charged sheet and suddenly start it moving, in the opposite direction with the same speed, only at the time T after we started the first current. The total current of the two added together is first zero, then on for a time T, then off again—because the two currents cancel. We have a square "pulse" of current.

The new negative current produces the same fields as the positive one, only with all the signs reversed and, of course, delayed in time by T. A wavefront again travels out at the velocity c. At the time t it has reached the distance $x = \pm c(t - T)$, as shown in Fig. 18-4(b). So we have two "blocks" of field marching out at the speed c, as in parts (a) and (b) of Fig. 18-4. The combined fields are as shown in part (c) of the figure. The fields are zero for $x > ct$, they are constant (with the values we found above) between $x = c(t - T)$ and $x = ct$, and again zero for $x < c(t - T)$.

In short, we have a little piece of field—a block of thickness cT—which has left the current sheet and is travelling through space all by itself. The fields have "taken off"; they are propagating freely through space, no longer connected in any way with the source. The caterpillar has turned into a butterfly!

How can this bundle of electric and magnetic fields maintain itself? The answer is: by the combined effects of the Faraday law, $\nabla \times E = -\partial B/\partial t$, and the new term of Maxwell, $c^2 \nabla \times B = \partial E/\partial t$. They cannot help maintaining themselves. Suppose the magnetic field were to disappear. There would be a changing magnetic field which would produce an electric field. If this electric field tries to go away, the changing electric field would create a magnetic field back again. So by a perpetual interplay—by the swishing back and forth from one field to the other—they must go on forever. It is impossible for them to disappear.* They maintain themselves in a kind of a dance—one making the other, the second making the first—propagating onward through space.

18-5 The speed of light

We have a wave which leaves the material source and goes outward at the velocity c, which is the speed of light. But let's go back a moment. From a historical point of view, it wasn't known that the coefficient c in Maxwell's equations was also the speed of light propagation. There was just a constant in the equations. We have called it c from the beginning, because we knew what it would turn out to be. We didn't think it would be sensible to make you learn the formulas with a different constant and then go back to substitute c wherever it belonged. From the point of view of electricity and magnetism, however, we just start out with two constants, ϵ_0 and c^2, that appear in the equations of electrostatics and magnetostatics:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \qquad (18.14)$$

and

$$\nabla \times B = \frac{j}{\epsilon_0 c^2}. \qquad (18.15)$$

If we take any *arbitrary* definition of a unit of charge, we can determine experimentally the constant ϵ_0 required in Eq. (18.14)—say by measuring the force between two unit charges at rest, using Coulomb's law. We must also determine experimentally the constant $\epsilon_0 c^2$ that appears in Eq. (18.15), which we can do, say, by measuring the force between two unit currents. (A unit current means one unit of charge per second.) The ratio of these two experimental constants is c^2—just another "electromagnetic constant."

---

* Well, not quite. They can be "absorbed" if they get to a region where there are charges. By which we mean that other fields can be produced somewhere which superpose on these fields and "cancel" them by destructive interference (see Chapter 31, Vol. I).

Notice now that this constant c^2 is the same no matter what we choose for our unit of charge. If we put twice as much "charge"—say twice as many proton charges—in our "unit" of charge, ϵ_0 would need to be one-fourth as large. When we pass two of these "unit" currents through two wires, there will be twice as much "charge" per second in each wire, so the force between two wires is four times larger. The constant $\epsilon_0 c^2$ must be reduced by one-fourth. But the ratio $\epsilon_0 c^2/\epsilon_0$ is unchanged.

So just by experiments with charges and currents we find a number c^2 which turns out to be the square of the velocity of propagation of electromagnetic influences. From static measurements—by measuring the forces between two unit charges and between two unit currents—we find that $c = 3.00 \times 10^8$ meters/sec. When Maxwell first made this calculation with his equations, he said that bundles of electric and magnetic fields should be propagated at this speed. He also remarked on the mysterious coincidence that this was the same as the speed of light. "We can scarcely avoid the inference," said Maxwell, "that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena."

Maxwell had made one of the great unifications of physics. Before his time, there was light, and there was electricity and magnetism. The latter two had been unified by the experimental work of Faraday, Oersted, and Ampere. Then, all of a sudden, light was no longer "something else," but was only electricity and magnetism in this new form—little pieces of electric and magnetic fields which propagate through space on their own.

We have called your attention to some characteristics of this special solution, which turn out to be true, however, for *any* electromagnetic wave: that the magnetic field is perpendicular to the direction of motion of the wavefront; that the electric field is likewise perpendicular to the direction of motion of the wavefront; and that the two vectors E and B are perpendicular to each other. Furthermore, the magnitude of the electric field E is equal to c times the magnitude of the magnetic field B. These three facts—that the two fields are transverse to the direction of propagation, that B is perpendicular to E, and that $E = cB$—are generally true for any electromagnetic wave. Our special case is a good one—it shows all the main features of electromagnetic waves.

18–6 Solving Maxwell's equations; the potentials and the wave equation

Now we would like to do something mathematical; we want to write Maxwell's equations in a simpler form. You may consider that we are complicating them, but if you will be patient a little bit, they will suddenly come out simpler. Although by this time you are thoroughly used to each of the Maxwell equations, there are many pieces that must all be put together. That's what we want to do.

We begin with $\nabla \cdot B = 0$—the simplest of the equations. We know that it implies that B is the curl of something. So, if we write

$$B = \nabla \times A, \tag{18.16}$$

we have already solved one of Maxwell's equations. (Incidentally, you appreciate that it remains true that another vector A' would be just as good if $A' = A + \nabla\psi$ —where ψ is any scalar field—because the curl of $\nabla\psi$ is zero, and B is still the same. We have talked about that before.)

We take next the Faraday law, $\nabla \times E = -\partial B/\partial t$, because it doesn't involve any currents or charges. If we write B as $\nabla \times A$ and differentiate with respect to t, we can write Faraday's law in the form

$$\nabla \times E = -\frac{\partial}{\partial t} \nabla \times A.$$

Since we can differentiate either with respect to time or to space first, we can also write this equation as

$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0. \tag{18.17}$$

We see that $E + \partial A/\partial t$ is a vector whose curl is equal to zero. Therefore that vector is the gradient of something. When we worked on electrostatics, we had $\nabla \times E = 0$, and then we decided that E itself was the gradient of something. We took it to be the gradient of $-\phi$ (the minus for technical convenience). We do the same thing for $E + \partial A/\partial t$; we set

$$E + \frac{\partial A}{\partial t} = -\nabla\phi. \tag{18.18}$$

We use the same symbol ϕ so that, in the electrostatic case where nothing changes with time and the $\partial A/\partial t$ term disappears, E will be our old $-\nabla\phi$. So Faraday's equation can be put in the form

$$E = -\nabla\phi - \frac{\partial A}{\partial t}. \tag{18.19}$$

We have solved two of Maxwell's equations already, and we have found that to describe the electromagnetic fields E and B, we need four potential functions: a scalar potential ϕ and a vector potential A, which is, of course, three functions.

Now that A determines part of E, as well as B, what happens when we change A to $A' = A + \nabla\psi$? In general, E would change if we didn't take some special precaution. We can, however, still allow A to be changed in this way without affecting the fields E and B—that is, without changing the physics—if we always change A and ϕ *together* by the rules

$$A' = A + \nabla\psi, \qquad \phi' = \phi - \frac{\partial\psi}{\partial t}. \tag{18.20}$$

Then neither B nor E, obtained from Eq. (18.19), is changed.

Previously, we chose to make $\nabla \cdot A = 0$, to make the equations of statics somewhat simpler. We are not going to do that now; we are going to make a different choice. But we'll wait a bit before saying what the choice is, because later it will be clear *why* the choice is made.

Now we return to the two remaining Maxwell equations which will give us relations between the potentials and the sources ρ and j. Once we can determine A and ϕ from the currents and charges, we can always get E and B from Eqs. (18.16) and (18.19), so we will have another form of Maxwell's equations.

We begin by substituting Eq. (18.19) into $\nabla \cdot E = \rho/\epsilon_0$; we get

$$\nabla \cdot \left(-\nabla\phi - \frac{\partial A}{\partial t}\right) = \frac{\rho}{\epsilon_0},$$

which we can write also as

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot A = \frac{\rho}{\epsilon_0}. \tag{18.21}$$

This is one equation relating ϕ and A to the sources.

Our final equation will be the most complicated. We start by rewriting the fourth Maxwell equation as

$$c^2\nabla \times B - \frac{\partial E}{\partial t} = \frac{j}{\epsilon_0},$$

and then substitute for B and E in terms of the potentials, using Eqs. (18.16) and (18.19):

$$c^2\nabla \times (\nabla \times A) - \frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial A}{\partial t}\right) = \frac{j}{\epsilon_0}.$$

The first term can be rewritten using the algebraic identity: $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$; we get

$$-c^2\nabla^2 A + c^2\nabla(\nabla \cdot A) + \frac{\partial}{\partial t}\nabla\phi + \frac{\partial^2 A}{\partial t^2} = \frac{j}{\epsilon_0}. \tag{18.22}$$

It's not very simple!

Fortunately, we can now make use of our freedom to choose arbitrarily the divergence of A. What we are going to do is to use our choice to fix things so that the equations for A and for ϕ are separated but have the same form. We can do this by taking*

$$\nabla \cdot A = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}. \tag{18.23}$$

When we do that, the two middle terms in A and ϕ in Eq. (18.22) cancel, and that equation becomes much simpler:

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{j}{\epsilon_0 c^2}. \tag{18.24}$$

And our equation for ϕ—Eq. (18.21)—takes on the same form:

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}. \tag{18.25}$$

What a beautiful set of equations! They are beautiful, first, because they are nicely separated—with the charge density, goes ϕ; with the current, goes A. Furthermore, although the left side looks a little funny—a Laplacian together with a $(\partial/\partial t)^2$—when we unfold it we see

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}. \tag{18.26}$$

It has a nice symmetry in x, y, z, t—the $-1/c^2$ is necessary because, of course, time and space *are* different; they have different units.

Maxwell's equations have led us to a new kind of equation for the potentials ϕ and A but to the same mathematical form for all four functions ϕ, A_x, A_y, and A_z. Once we learn how to solve these equations, we can get B and E from $\nabla \times A$ and $-\nabla\phi - \partial A/\partial t$. We have another form of the electromagnetic laws exactly equivalent to Maxwell's equations, and in many situations they are much simpler to handle.

We have, in fact, already solved an equation much like Eq. (18.26). When we studied sound in Chapter 47 of Vol. I, we had an equation of the form

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2},$$

and we saw that it described the propagation of waves in the x-direction at the speed c. Equation (18.26) is the corresponding wave equation for three dimensions. So in regions where there are no longer any charges and currents, the solution of these equations is *not* that ϕ and A are zero. (Although that is indeed one possible solution.) There are solutions in which there is some set of ϕ and A which are changing in time but always moving out at the speed c. The fields travel onward through free space, as in our example at the beginning of the chapter.

With Maxwell's new term in Eq. IV, we have been able to write the field equations in terms of A and ϕ in a form that is simple and that makes immediately apparent that there are electromagnetic waves. For many practical purposes, it will still be convenient to use the original equations in terms of E and B. But they are on the other side of the mountain we have already climbed. Now we are ready to cross over to the other side of the peak. Things will look different—we are ready for some new and beautiful views.

---

* Choosing the $\nabla \cdot A$ is called "choosing a gauge." Changing A by adding $\nabla\psi$ is called a "gauge transformation." Equation (18.23) is called "the Lorentz gauge."